Characterizing persistent homology via 0-dimensional resolutions*

Marco Delgado-Garrido[†1], Alvaro Torras-Casas[‡2], and Rocio Gonzalez-Diaz[§1]

[1]Depto. de Matemática Aplicada I, Universidad de Sevilla, Seville, Spain
[2]Inserm, INRAE, CRESS, Université Paris Cité and Université Sorbonne Paris Nord, France

Abstract

We study how the p-dimensional persistent homology module from the Vietoris-Rips filtration of a point cloud X can be determined by the 0-dimensional persistent homology of *inverse barcode problem* solutions, via a specific persistence module resolution. We also discuss how resolutions of 0-dimensional persistent homology modules given by a presentation preserve 0-dimensional additive partial matchings and how we can bound p-dimensional partial matchings in terms of the well-behaved 0-dimensional ones. These results can be seen as a first step towards a proof of the stability of additive partial matchings.

1 Introduction

Persistent homology has become one of the most powerful tools in Topological Data Analysis (TDA) [1]. The categorical-algebraic perspective provided by persistence modules helps us understand and develop the theory of persistent homology [2].

In practice, the TDA pipeline [3] consists of taking a point cloud X, building a specific filtration of the data (e.g., Vietoris-Rips), extracting p-dimensional topological-scale information by computing the persistent homology modules $\mathrm{PH}_p(X)$ using simplicial homology over $\mathbb{Z}_2$, and visualizing barcodes $\mathcal{B}(\mathrm{PH}_p(X))$ using specialized software [4].

A persistence module $M : \mathbb{R} \to \mathbf{Vec}_k$ is a functor from the real numbers to the category of vector spaces over a fixed field k. The vector space $M(t) \in \mathbf{Vec}_k$ is denoted by M_t. The direct sum, intersection, and quotient of persistence modules are also persistence modules [5]. An interval $I \subset \mathbb{R}$ is considered of the form $I = \langle a, b\rangle$ for $a \leq b \in \mathbb{R}$, or $b = +\infty$. Given an interval $I \subset \mathbb{R}$, the *interval module* k_I is the persistence module where $k_{It} = k$ for all $t \in I$ and $k_{It} = 0$ otherwise, while the structure maps $k_{It} \to k_{Is}$, with $t \leq s$, are the identity whenever possible, and zero otherwise. When the persistence module is pointwise finite-dimensional (p.f.d.) then it can be decomposed uniquely (up to isomorphism) as a direct sum of interval modules, and are completely described by a multiset called its *barcode* (see Th. 1).

Given two persistence modules U and V, a *persistence morphism* $f : V \to U$ is a natural transformation i.e., a collection of linear maps $\{f_c\}_{c\in \mathrm{Obj}(\mathcal{C})}$ that commute with the *structure maps* of the persistence modules U and V. A question that arises is whether $f : V \to U$ induces a relation—specifically, a partial matching—between their barcodes. It is known that such partial matchings cannot be functorial [6]. Recently, a *block function* associated with the morphism $\mathcal{M}_f$ [7] was proposed, which is algebraically well-defined, linear with respect to the direct sum of persistence morphisms, efficiently computable via matrix reduction techniques, and induces a partial matching between the barcodes of U and V.

We establish in this paper that the p-dimensional persistent homology module is linked to the 0-dimensional persistent homology of a solution to the *inverse barcode problem* allowing us to determine the former from the latter, thanks to a specific persistence module resolution. This approach aligns with the recent idea in AI, particularly with Kolmogorov-Arnold Networks (KANs) [8], suggesting that the information of a high-dimensional point cloud is encoded in a composition of lower-dimensional structures. We show that certain resolutions of 0-dimensional persistent homology preserve additive partial matchings [9], which allows us to bound p-dimensional matchings using well-behaved 0-dimensional ones.

2 Persistence bases and presentations

Interval modules are the “building blocks” of persistence modules.

Theorem 1 (from [10]) *Every p.f.d. persistence module V can be decomposed uniquely:*

$$V \cong \bigoplus_{I\in S^V} \left(\bigoplus_{i=1}^{m_I} k_I\right) \text{ where } S^V \text{ is a set of intervals.}$$

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†Email: mardelgar11@alumn.us.es
‡Email: alvaro.torras-casas@inserm.fr
§Email: rogodi@us.es

The multiset $\mathcal{B}(V) = (S^V, m)$ is called the barcode of V where $m : S^V \to \mathbb{N}$ and $m(I)$ is denoted by m_I.

A *free persistence module* is of the form $\bigoplus_{i\in\Gamma} k_{[a_i,+\infty)}$ for an index set Γ and values $a_i \in \mathbb{R}$. A *persistence basis* for a persistence module V is an isomorphism

$$\alpha : \bigoplus_{i\in\Gamma} k_{\langle a_i,b_i\rangle} \to V,$$

where Γ is an index set and $\langle a_i, b_i\rangle$ denotes an interval. By Th. 1, a persistence basis exists for any p.f.d persistence module. The persistence generator $\alpha_i : k_{\langle a_i,b_i\rangle} \to V$ is defined as the morphism α restricted to $k_{\langle a_i,b_i\rangle}$ for $i \in \Gamma$. We can also specify a persistence basis α by its set of persistence generators, $\mathcal{A} = \{\alpha_i\}_{i\in\Gamma}$. Given a subset $S = \{\alpha_i\}_{i\in J} \subset \mathcal{A}$, we define the span of S, denoted by $\langle S\rangle$, as the image of the sum of the persistence generators of S, that is,

$$\langle S\rangle = \mathrm{Im}\Big(\bigoplus_{i\in J} \alpha_i : k_{\langle a_i,b_i\rangle} \to V\Big).$$

For $t \in \mathbb{R}$, we define $S_t := \{\alpha_{it}(1_k) : i \in J, t \in \langle a_i, b_i\rangle\}$. In particular, $V_t = \langle \mathcal{A}_t\rangle$ and $\langle S\rangle_t = \langle S_t\rangle$, being $\mathcal{A}_t$ and S_t linearly independent sets of vectors in V_t. Given $\mathcal{A}$ and $I = \langle a, b\rangle$, we define the sets of generators [9]:

$$\widetilde{\mathcal{A}}_I^+ = \{\alpha_i \in \mathcal{A} \mid (a_i < a) \text{ or } (a_i = a \text{ and } b_i \le b)\},$$
$$\widetilde{\mathcal{A}}_I^- = \{\alpha_i \in \mathcal{A} \mid (a_i < a) \text{ or } (a_i = a \text{ and } b_i < b)\},$$
$$\widehat{\mathcal{A}}_I^+ = \{\alpha_i \in \mathcal{A} \mid (b_i < b) \text{ or } (b_i = b \text{ and } a_i \le a)\},$$
$$\widehat{\mathcal{A}}_I^- = \{\alpha_i \in \mathcal{A} \mid (b_i < b) \text{ or } (b_i = b \text{ and } a_i < a)\}.$$

3 Block functions

Given two barcodes $B_1 = (S_1, m)$ and $B_2 = (S_2, n)$, a *block function* [7] is a function

$$\mathcal{M} : S_1 \times S_2 \longrightarrow \mathbb{N} \cup \{0\}$$

such that $\sum_{J\in S_2} \mathcal{M}(I, J) \le m_I$ for every $I \in S_1$. When $\mathcal{M}$ also satisfies that $\sum_{I\in S_1} \mathcal{M}(I, J) \le n_J$ for every $J \in S_2$ then $\mathcal{M}$ induces a partial matching between B_1 and B_2 (Remark 2.5 [7]).

Given a persistence morphism $f : V \to U$, a persistence basis $\mathcal{A}$ of V and a persistence basis $\mathcal{B}$ of U we define the vector space:

$$Z_{IJt} := \frac{f\tilde{V}_{It}^+ \cap \hat{U}_{Jt}^+}{f\tilde{V}_{It}^- \cap \hat{U}_{Jt}^+ + f\tilde{V}_{It}^+ \cap \hat{U}_{Jt}^-}$$

where, for all $t \in I$ and $s \in J$:

$$\widetilde{V}_{It}^{\pm} = \left\langle \widetilde{\mathcal{A}}_{It}^{\pm} \right\rangle, \qquad \widehat{V}_{It}^{\pm} = \left\langle \widehat{\mathcal{A}}_{It}^{\pm} \right\rangle,$$
$$\widetilde{U}_{Js}^{\pm} = \left\langle \widetilde{\mathcal{B}}_{Js}^{\pm} \right\rangle, \qquad \widehat{U}_{Js}^{\pm} = \left\langle \widehat{\mathcal{B}}_{Js}^{\pm} \right\rangle.$$

We define the following operator:

$$\widetilde{\mathcal{M}}_f(I, J) := \dim Z_{IJd} \text{ where } I = \langle a, b\rangle, J = \langle c, d\rangle.$$

Theorem 2 (from [9]) *The operator $\widetilde{\mathcal{M}}_f$ is a block function and always induces a unique partial matching between $\mathcal{B}(V)$ and $\mathcal{B}(U)$.*

4 0-Dimensional persistent homology resolutions and additive matchings

We restrict our study to persistence modules obtained from a point cloud $X \subset \mathbb{R}^n$ via the Vietoris-Rips filtration defined below.

Let X be a finite metric space (point cloud) and let $\varepsilon > 0$. The *Vietoris–Rips complex* $\mathrm{VR}_\varepsilon(X)$ associated with X at scale ε is the simplicial complex whose vertices are the points in X, and a set of points $x_0, \dots, x_p$ spans a p-simplex of $\mathrm{VR}_\varepsilon(X)$ if and only if

$$d(x_i, x_j) \le \varepsilon \quad \text{for all } 0 \le i, j \le p.$$

The *Vietoris-Rips filtration* of X is the family of Vietors-Rips complexes $\{\mathrm{VR}_\varepsilon(\mathrm{X})\}_{\varepsilon\in\mathbb{R}}$. See Fig. 1.

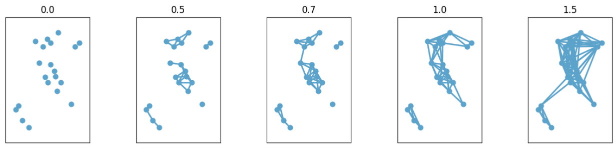


Figure 1: Left $\varepsilon = 0$: Point cloud. Visualization of the associated 1-skeleton of $\mathrm{VR}_\varepsilon(\mathrm{X})$ for some values of ε.

We denote the p-dimensional persistent homology of the Vietoris-Rips filtration of X by $\mathrm{PH}_p(X)$. It captures the *p-holes* at every value of ε such as connected components ($p = 0$), holes ($p = 1$), or cavities ($p = 2$).

Note that the inclusion of point clouds $X \hookrightarrow Y$ induces an inclusion $\mathrm{VR}_\varepsilon(X) \hookrightarrow \mathrm{VR}_\varepsilon(Y)$ for every ε which induces a persistence morphism $f : \mathrm{PH}_p(X) \to \mathrm{PH}_p(Y)$. So, to study topological features of subsets with respect to a dataset, we could study persistence morphisms induced by the inclusions.

Lemma 3 *Given point clouds X, Y, and a persistence morphism $f : \mathrm{PH}_0(X) \to \mathrm{PH}_0(Y)$ (not necessarily induced by the inclusion), there are free persistence modules, M_0, M_0', M_1, M_1', and persistence morphisms $f_0 : M_0 \to M_0'$, $f_1 : M_1 \to M_1'$, such that:*

$$\begin{tikzcd}
0 \arrow[r] & M_1 \arrow[r, hook] \arrow[d, "f_1"] & M_0 \arrow[r, two heads] \arrow[d, "f_0"] & \mathrm{PH}_0(X) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & M_1' \arrow[r, hook] & M_0' \arrow[r, two heads] & \mathrm{PH}_0(Y) \arrow[r] & 0
\end{tikzcd}$$

where the rows are short exact sequences (i.e. resolutions). The matrix representation of f in terms of persistence bases equals that of f_0; the matrix representation of f_1 is the result of removing the column and row associated with the infinite persistence interval.

Sketch of the proof: The following commutative diagram is obtained by applying Th. 1 to $\mathrm{PH}_0(X)$ and $\mathrm{PH}_0(Y)$ and building M_0 and M_1 in $O(1)$ time given the $\mathrm{PH}_0(X)$ decomposition. Knowing that every interval module in the decomposition of $\mathrm{PH}_0(X)$ and $\mathrm{PH}_0(Y)$ starts at 0, one can explicitly build the free modules M_0, M_0', M_1 and M_1' as follows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus\limits_{[0,b)\in S^X} \big(\bigoplus\limits_{m_{[0,b)}} k_{[b,+\infty)}\big) \arrow[r, hook] \arrow[d, "f_1"] & \bigoplus\limits_{i=1}^{\sum_{[0,b)\in S^X} m_{[0,b)}} k_{[0,+\infty)} \arrow[r, two heads] \arrow[d, "f_0"] & \bigoplus\limits_{[0,b)\in S^X} \big(\bigoplus\limits_{m_{[0,b)}} k_{[0,b)}\big) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \bigoplus\limits_{[0,\beta)\in S^Y} \big(\bigoplus\limits_{m_{[0,\beta)}} k_{[\beta,+\infty)}\big) \arrow[r, hook] & \bigoplus\limits_{i=1}^{\sum_{[0,\beta)\in S^Y} m_{[0,\beta)}} k_{[0,+\infty)} \arrow[r, two heads] & \bigoplus\limits_{[0,\beta)\in S^Y} \big(\bigoplus\limits_{m_{[0,\beta)}} k_{[0,\beta)}\big) \arrow[r] & 0
\end{tikzcd}$$

Knowing that each interval module in the decomposition of X (resp. Y) can be uniquely associated with the equivalence class of a point of X (resp. Y), we can build persistence bases in such a way that the diagram commutes and the rows are exact. See Fig 2. $\square$

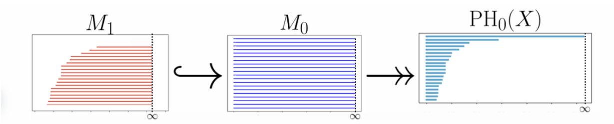


Figure 2: Visual description of the short exact sequence: $0 \to M_1 \hookrightarrow M_0 \twoheadrightarrow \mathrm{PH}_0(X) \to 0$ associated with the point cloud from Fig. 1. Intervals are ordered such that the quotient of an interval of $\mathcal{B}(M_0)$ by the interval of $\mathcal{B}(M_1)$ at the same position results in the interval of $\mathcal{B}(\mathrm{PH}_0(X))$ at the same position too.

Lemma 4 *Under the hypothesis of Lemma 3 and choosing an adequate persistence basis for each persistence module,*

$$\widetilde{\mathcal{M}}_f([0,b),[0,\beta)) = \widetilde{\mathcal{M}}_{f_0}([b,+\infty),[\beta,+\infty))$$

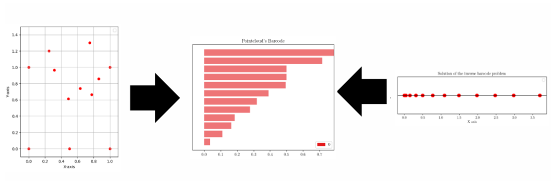
where $b, \beta \in \mathbb{R}_{\geq 0}$.

Sketch of the proof: The vector spaces $Z_{[0,b)\;[0,\beta)\;\beta}$ and $Z_{[b,+\infty)\;[\beta,+\infty)\;d}$ have the same dimension with d big enough. $\square$

5 Main Results

Lemma 5 (inverse barcode problem.) *Let $\mathcal{B}(V)$ be the barcode of a persistence module V such that: every interval starts at 0; there is a finite number of intervals; and there is exactly one interval ending at $+\infty$. Then, it is possible to build a point cloud $A \subset \mathbb{R}$ such that $\mathcal{B}(V) = \mathcal{B}(\mathrm{PH}_0(A))$, i.e, $\mathrm{PH}_0(A) \cong V$, where A is a solution to the inverse barcode problem of V.*

Sketch of the proof: We set $0 \in A$, corresponding to the infinite persistence interval of the barcode, and then proceed by adding, to A, the points on the real line corresponding to the cumulative sum of the right endpoints. See the figure below. $\square$



Lemma 6 *Given point clouds X, Y, and a persistence morphism $g : \mathrm{PH}_p(X) \to \mathrm{PH}_p(Y)$ (not necessarily induced by the inclusion), there exist free persistent modules, M_0, M_0', M_1, M_1', and persistence morphisms $f_0 : M_0 \to M_0'$, $f_1 : M_1 \to M_1'$, such that the diagram of Fig. 3 commutes and:*

- *$0 \to M_1 \hookrightarrow M_0 \twoheadrightarrow \mathrm{PH}_p(X) \to 0$ is a resolution (idem for Y).*
- *$0 \to M_i \hookrightarrow N_i \twoheadrightarrow \mathrm{PH}_0(X_i^p) \to 0$, $i = 0, 1$, is a resolution to which we apply Lemma 3, and X_i^p is the inverse barcode problem solution for the quotient of free modules N_i/M_i (idem for Y).*
- *Applying Lemma 3, the matrix representation of f_i is the same as the matrix representation of γ_i and g_i (removing any column and row associated with infinite intervals) and the additive partial matching of f_i is the same as the one of g_i in the sense of Lemma 4 (the same holds for Y).*

Sketch of the proof: By Th. 1, we can decompose the persistence modules $\mathrm{PH}_p(X)$ and $\mathrm{PH}_p(Y)$, and then obtain the commutative rectangle in the center of the diagram of Fig. 3, building M_0, M_0', M_1 and M_1' in a similar fashion as Lemma 3. The outer structure in the diagram follows from considering the free persistence modules M_0, M_0', M_1, M_1' as the first module (starting from the left) in the resolution of one of the rows in Lemma 3. $\square$

The following result characterizes the p-dimensional persistent homology of a point cloud X in terms of the 0-dimensional homology of certain inverse barcode problem solutions associated with the p-dimensional barcode with respect to the persistence morphism of persistent homology modules.

Theorem 7 (Characterization of PH_p) *Under the hypothesis of Lemma 3, we have the following commutative diagram:*

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{PH}_0(X_0^p) \arrow[r, hook] \arrow[d, "g_0"] & \mathrm{PH}_0(X_1^p) \arrow[r, two heads] \arrow[d, "g_1"] & \mathrm{PH}_p(X) \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & \mathrm{PH}_0(Y_0^p) \arrow[r, hook] & \mathrm{PH}_0(Y_1^p) \arrow[r, two heads] & \mathrm{PH}_p(Y) \arrow[r] & 0
\end{tikzcd}$$

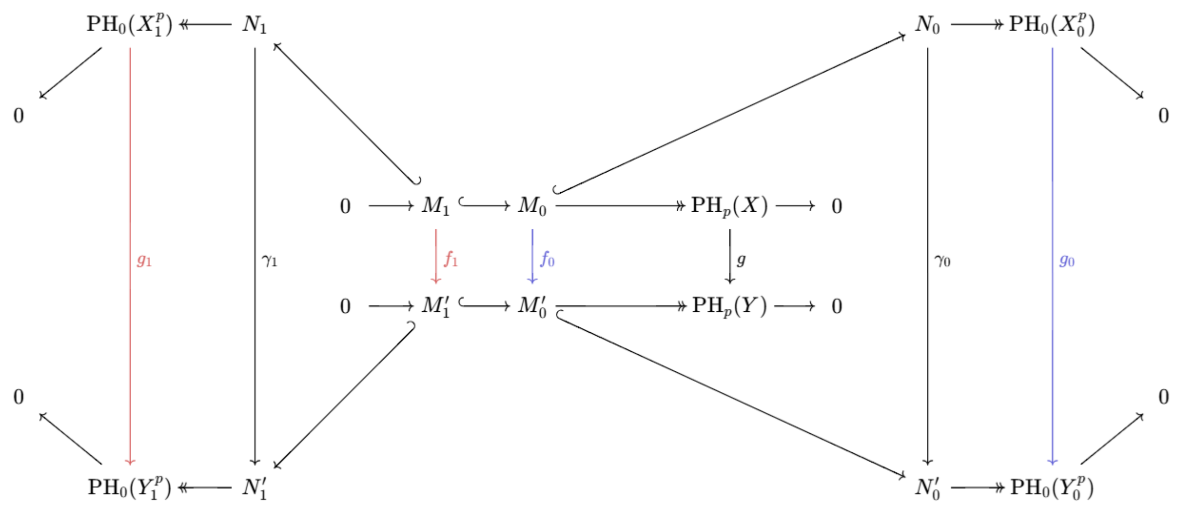
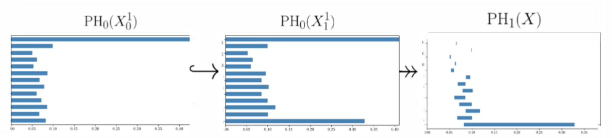


Figure 3: Diagram for Lemma 6.

Sketch of the proof: Intervals of $\mathrm{PH}_0(X_1^p)$ start at 0 and end at the right endpoint of the associated interval of $\mathrm{PH}_p(X)$, and intervals of $\mathrm{PH}_0(X_0^p)$ start at 0 and end at the left endpoint of the associated interval of $\mathrm{PH}_p(X)$. The same holds for $\mathrm{PH}_p(Y)$. We deduce that the diagram commutes and the rows are exact. $\square$



A direct result of the previous one is:

$$\mathrm{PH}_p(X) \cong \frac{\mathrm{PH}_0(X_1^p)}{\mathrm{PH}_0(X_0^p)}, \qquad \mathrm{PH}_p(Y) \cong \frac{\mathrm{PH}_0(Y_1^p)}{\mathrm{PH}_0(Y_0^p)}.$$

Theorem 8 *Under the hypothesis of Lemma 3, if $I = [a, b)$ and $J = [\alpha, \beta)$, $a, b, \alpha, \beta \in \mathbb{R}_{\geq 0}$, then:*

$$\begin{aligned}\widetilde{\mathcal{M}}_g(I, J) &\leq \widetilde{\mathcal{M}}_{f_0}([a, +\infty), [\alpha, +\infty)) \\ &\quad + \widetilde{\mathcal{M}}_{f_1}([b, +\infty), [\beta, +\infty)) \\ &= \widetilde{\mathcal{M}}_{g_0}([a, +\infty), [\alpha, +\infty)) \\ &\quad + \widetilde{\mathcal{M}}_{g_1}([b, +\infty), [\beta, +\infty)).\end{aligned}$$

Sketch of the proof: We get the bound by examining the dimension of the vector spaces Z_{IJt}. $\square$

6 Future work and open problems

We plan to give a complete description of $\widetilde{\mathcal{M}}_g([a, b), [\alpha, \beta))$ in terms of $\widetilde{\mathcal{M}}_{g_0}([a, +\infty), [\alpha, +\infty))$ and $\widetilde{\mathcal{M}}_{g_1}([b, +\infty), [\beta, +\infty))$. We also plan to apply these results to data quality and to the study of persistence bimodules via fibered barcodes.

Code availability: Code and examples available at https://github.com/Cimagroup/tdqual.git.

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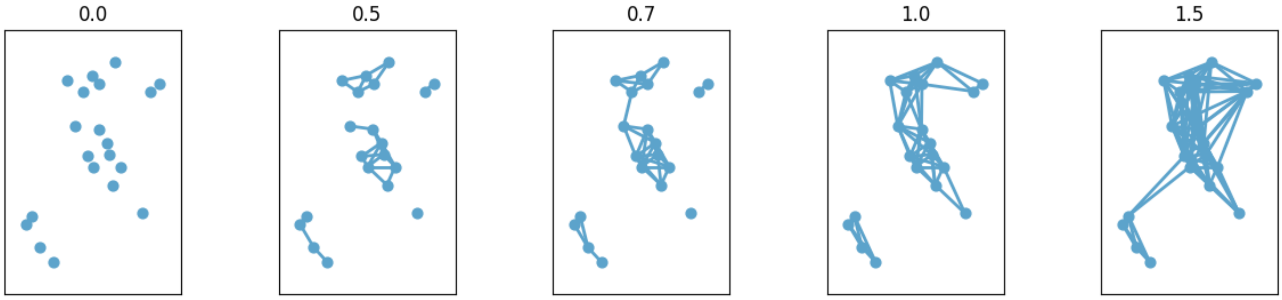
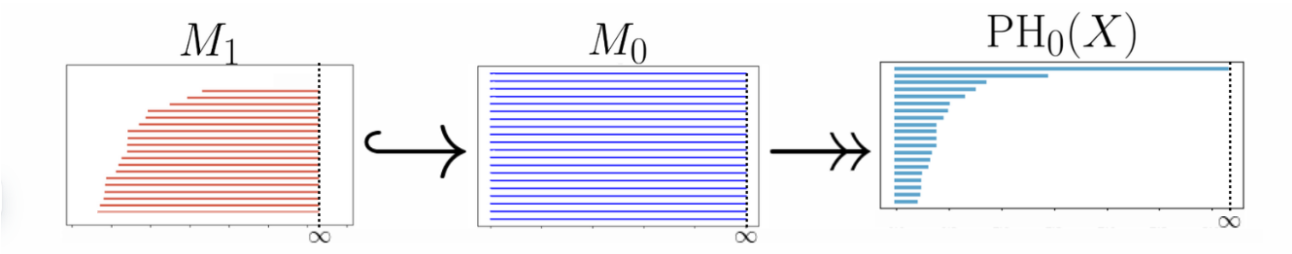
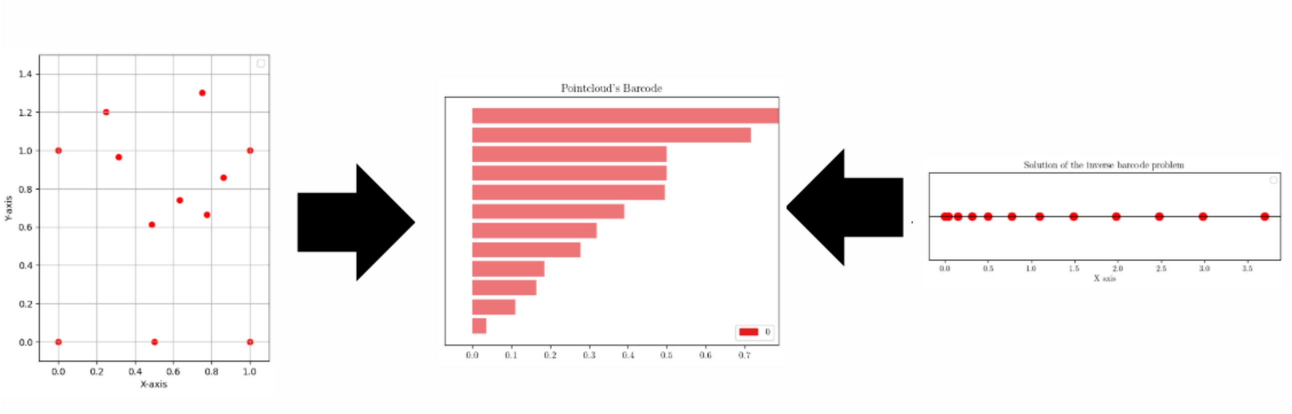
Appendix. Figures and complementary example: In this appendix, we present the figures in the same order but in a larger format to enhance readability and provide greater clarity for the reader, and an example of application of the theory developed in the paper with a link to the code.

A Figures:







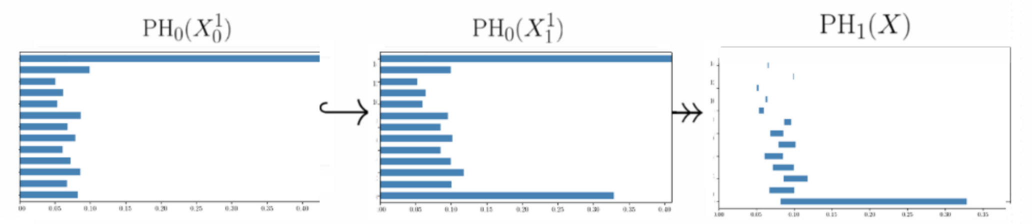


Figure 4: Characterization of $\mathrm{PH}_1(X)$.

$$
\begin{tikzcd}
0 \arrow[r] & \bigoplus_{[0,b)\in S^X} \left(\bigoplus_{m_{[0,b)}} k_{[b,+\infty)} \right) \arrow[r, hook] \arrow[d, "f_1"] & \bigoplus_{i=1}^{\sum_{[0,b)\in S^X} m_{[0,b)}} k_{[0,+\infty)} \arrow[r, two heads] \arrow[d, "f_0"] & \bigoplus_{[0,b)\in S^X} \left(\bigoplus_{m_{[0,b)}} k_{[0,b)} \right) \arrow[r] \arrow[d, hook, "f"] & 0 \\
0 \arrow[r] & \bigoplus_{[0,\beta)\in S^Y} \left(\bigoplus_{m_{[0,\beta)}} k_{[\beta,+\infty)} \right) \arrow[r, hook] & \bigoplus_{i=1}^{\sum_{[0,\beta)\in S^Y} m_{[0,\beta)}} k_{[0,+\infty)} \arrow[r, two heads] & \bigoplus_{[0,\beta)\in S^Y} \left(\bigoplus_{m_{[0,\beta)}} k_{[0,\beta)} \right) \arrow[r] & 0
\end{tikzcd}
$$

Figure 5: Diagram for the proof of Lemma 3

$$
\begin{tikzcd}
 & \mathrm{PH}_0(X_1^p) \arrow[dl] \arrow[ddddd, red, "g_1"] & N_1 \arrow[l, two heads] \arrow[ddddd, "\gamma_1"] & & & & & & N_0 \arrow[r, two heads] \arrow[ddddd, "\gamma_0"] & \mathrm{PH}_0(X_0^p) \arrow[dr] \arrow[ddddd, blue, "g_0"] & \\
0 & & & & & & & & & & 0 \\
 & & & 0 \arrow[r] & M_1 \arrow[r, hook] \arrow[uull, hook] \arrow[d, red, "f_1"] & M_0 \arrow[r, two heads] \arrow[uurrr, hook] \arrow[d, blue, "f_0"] & \mathrm{PH}_p(X) \arrow[r] \arrow[d, "g"] & 0 & & & \\
 & & & 0 \arrow[r] & M_1' \arrow[r, hook] \arrow[ddll, hook] & M_0' \arrow[r, two heads] \arrow[ddrrr, hook] & \mathrm{PH}_p(Y) \arrow[r] & 0 & & & \\
0 & & & & & & & & & & 0 \\
 & \mathrm{PH}_0(Y_1^p) \arrow[ul] & N_1' \arrow[l, two heads] & & & & & & N_0' \arrow[r, two heads] & \mathrm{PH}_0(Y_0^p) \arrow[ur] &
\end{tikzcd}
$$

Figure 6: Diagram for Lemma 6.

B Noisy circle example

Code available at: https://github.com/Cimagroup/tdqual.git

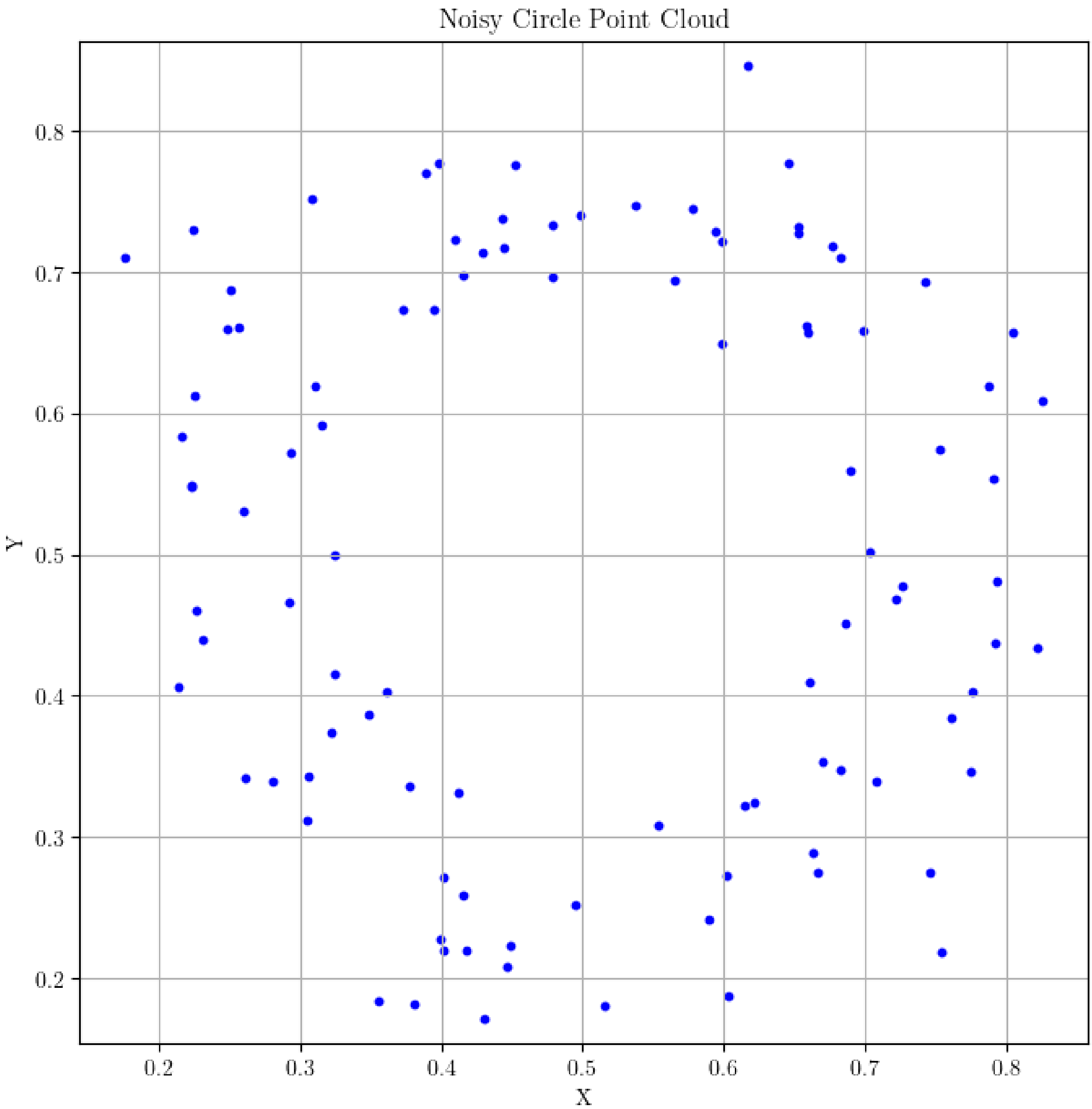


Figure 7: Noisy circle point cloud X.

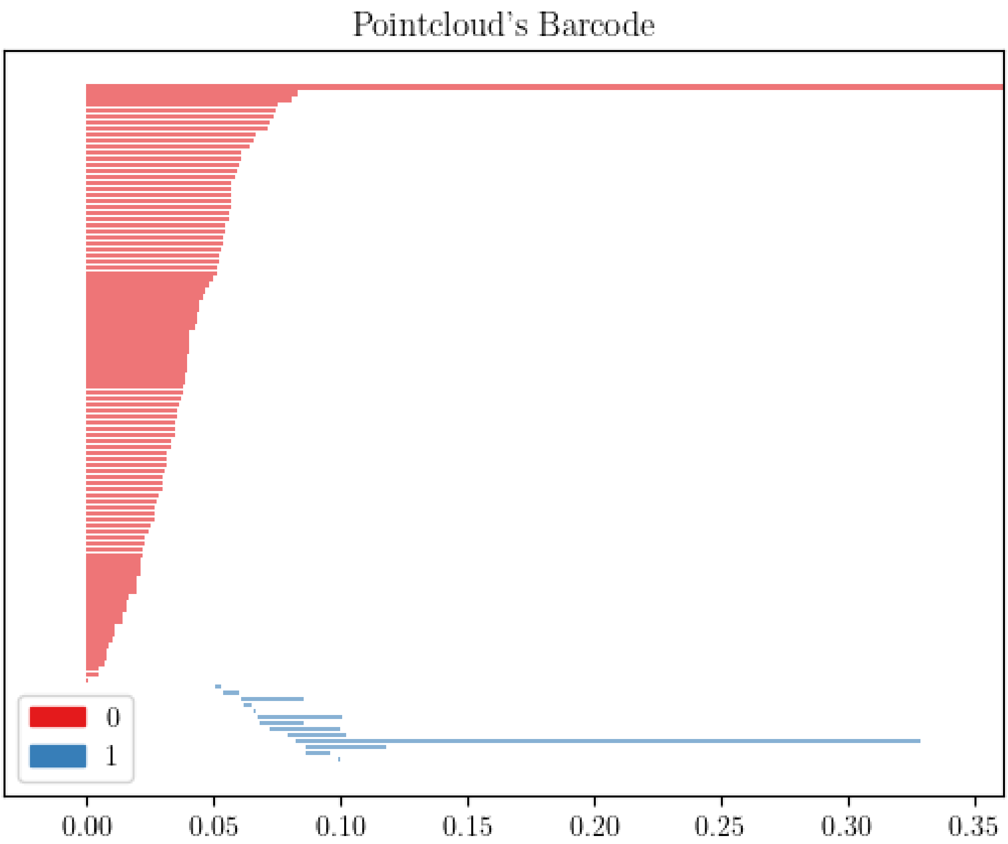


Figure 8: 0-dimensional and 1-dimensional barcode of X.

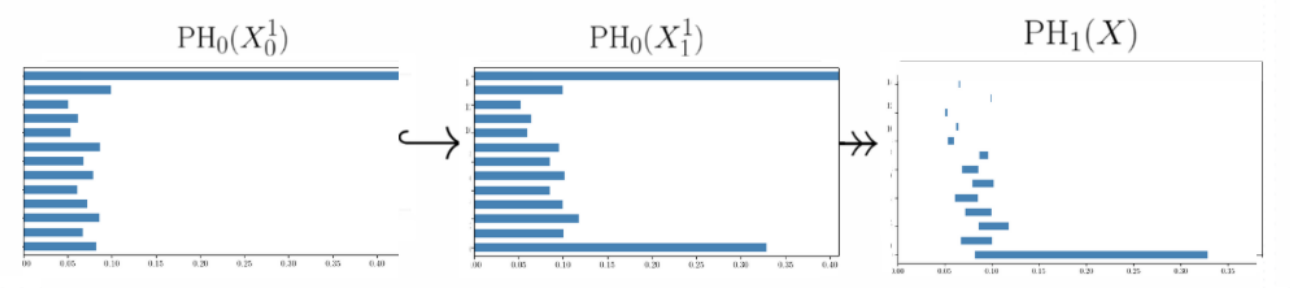


Figure 9: Characterization of $\mathrm{PH}_1(X)$.

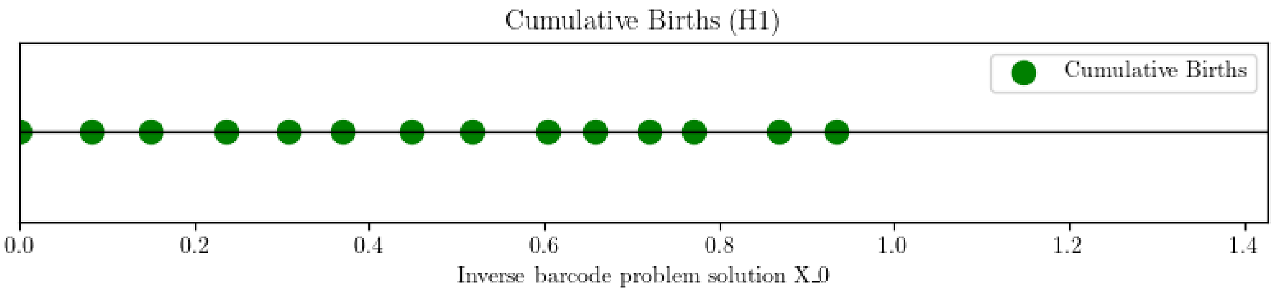


Figure 10: Inverse barcode problem solution X_0^1.

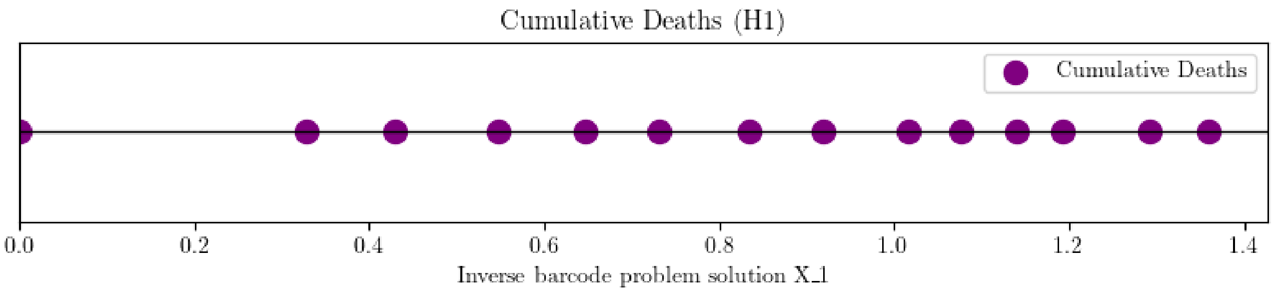


Figure 11: Inverse barcode problem solution X_1^1.