On the number of quadrilaterals in higher order Voronoi diagrams*

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Given a point set S in general position in the plane, the Voronoi diagram of order k of S, $V_k(S)$, is the subdivision of the plane into faces such that points in the same face have the same set of k nearest points of S. It is well known that for $k \geq 2$, $V_k(S)$ cannot contain any triangular face. For quadrilateral faces, it is shown in [2] that there are point sets S such that $V_k(S)$ has no quadrilaterals (for several values of k), and point sets S such that $V_k(S)$ only contains one quadrilateral for any value of k. In this paper, we focus on upper bounds for the number of quadrilaterals, $q_k(S)$, that $V_k(S)$ can contain.

Given a set S of N points, a face f of $V_k(S)$ is denoted by $f(P_k)$, where P_k is the subset of k points of S that is closest to every point of this face. Each vertex of a face $f(P_k)$ is either the circumcenter of two points from P_k and one point from $S \setminus P_k$, a type II vertex, or one point from P_k and two points from $S \setminus P_k$, a type I vertex. It is known that the number of circles through three points of S that enclose k other points of S is $c_k = (k+1)(2N-k-2) - \sum_{j=0}^{k} e_j$, where e_j denotes the number of j-edges of S (a j-edge is a half-plane defined by the oriented line through a pair of points of S that contains j points of S in its interior). Then, the number of vertices of type I is c_{k-1} and the number of vertices of type II is c_{k-2} [1].

A quadrilateral in $V_k(S)$ always consists of two vertices of type I and two vertices of type II, alternating along the boundary of the quadrilateral [1]. Besides, in [2] the authors prove that two quadrilaterals cannot be adjacent in $V_k(S)$. Using these properties, we can prove the following theorem.

Theorem 1 *Let S be a set of N points in general position in the plane. Then,* $q_k(S) \leq \min\left\{\frac{2kN-2N-k^2+k-\sum_{j=0}^{k-2} e_j}{2}, \frac{2kN-k^2-k-\sum_{j=0}^{k-1} e_j}{2}\right\}$

When $k=2$, it is easy to check that for any point set S, the upper bound given in the previous theorem is $N-1-\frac{e_0}{2}$, where e_0 is the number of points on the boundary of the convex hull of S. In some cases, this upper bound is tight, as shown in Theorem 2. If the points of S are in convex position, the previous upper bound is $\frac{N}{2}-1$, which is also tight (see Theorem 3).

Theorem 2 *There exist sets S of N points in general position in the plane such that $q_2(S) = N-1-\lceil \frac{e_0}{2} \rceil$.*

For any values of N and k, we can build point sets S such that $V_k(S)$ has quite a few quadrilaterals.

Theorem 3 *For k even, there exist sets S of N points in general position in the plane (and also in convex position) such that for $k \leq \frac{N}{2}$, $q_k(S) \geq \frac{(2N-3k+1)\cdot(k-1)+1}{4}$, and for $k > \frac{N}{2}$, $q_k(S) \geq \frac{(N-k)^2}{4}$. If k is odd, then $q_k(S) \geq \frac{(2N-3k+1)\cdot(k-1)}{4}$ for $k \leq \frac{N}{2}$, and $q_k(S) \geq \frac{(N-k)^2-1}{4}$ for $k > \frac{N}{2}$.*

For point sets S in convex position, let $q_k(N) = \max_{|S|=N}\{q_k(S)\}$. Since $e_j = N$ for any j, we have the following result from Theorem 1.

Corollary 1 *If $k \leq \frac{N}{2}$, then $q_k(N) \leq \frac{(k-1)(N-k)}{2}$, and if $k > \frac{N}{2}$, then $q_k(N) \leq \frac{k(N-k-1)}{2}$.*

In some cases, the upper bound $\frac{(k-1)(N-k)}{2}$ for $q_k(N)$ is tight, as next Theorem 4 shows for $k = \frac{N}{2}$. However, the tightness of the bound is not true in general. For instance, if $N=7$ and $k=3$, then $\frac{(k-1)(N-k)}{2} = 4$, but $q_3(7) = 3$ (see Theorem 5).

Theorem 4 *There exist sets S of $N = 2n \geq 6$ points in the plane in convex position such that $q_n(S) = \frac{(n-1)n}{2}$.*

Theorem 5 *For $N=7$, $q_3(N) = 3$.*

References

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