Polytope depth bounds from convex hull decompositions

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The *depth* of a polytope refers to the minimum number of alternations between Minkowski sums and convex hulls required to construct the polytope. This notion captures, in this sense, how "complex" a polytope is to compute [1].

We define this concept recursively and denote it as $d(P)$ for a polytope P. If P consists of a single point, we define $d(P) = 0$. For any other polytope, we define $d(P) = m$, where m is the smallest positive integer such that

$$P = \sum_{i=1}^{p} \operatorname{conv}(P_{i1}, P_{i2}), \quad \text{with } d(P_{ij}) < m \quad \forall i, j. \tag{1}$$

This concept is particularly relevant due to its connection with a conjecture on the representability of ReLU neural networks [2], a problem that has only been solved under restrictive assumptions [3, 4]. One of the main challenges in resolving the conjecture in [2] is the lack of a characterization for the set of polytopes with a given depth. Currently, only polytopes of depth 1 are fully characterized, which are zonotopes.

To advance in this direction, we generalize a tool from [1] that was used to derive their main results. Specifically, if $P = \operatorname{conv}(x_1, \ldots, x_k)$, where x_i are the vertices of P, then

$$d(P) \leq \lceil \log_2 k \rceil. \tag{2}$$

We generalize this by providing a strategy to bound the depth of a polytope P, given a decomposition $P = \operatorname{conv}(P_1, \ldots, P_k)$ and known depth bounds for each polytope P_i.

The strategy iteratively applies the following bound:

$$d(\operatorname{conv}(P_i, P_j)) \leq \max\{m_i, m_j\}+1, \text{ where } d(P_i) \leq m_i,$$

selecting a pair (P_i, P_j) at each step. This procedure can be modeled as a full binary tree over the natural numbers with a binary operation $(\mathbb{N}, *)$, defined by $x * y = \max\{x, y\} + 1$. The goal is to construct a tree that yields the smallest possible depth bound for P.

We show that a greedy strategy–always choosing the pair of polytopes with the smallest depth bound at each step–results in an optimal tree. This leads to an optimal bound on the depth of P when considering only convex hull operations and disregarding Minkowski sums in (1). As a consequence of this greedy strategy, we obtain the following results.

Theorem 1 *If $P = \operatorname{conv}(P_1, \ldots, P_k)$ and $d(P_i) \leq m$ for all i, then*

$$d(P) \leq \lceil \log_2 k \rceil + m.$$

We then generalize (2) and derive analogous bounds for edges and 2-faces.

Theorem 2 *Let $f = (f_0, \ldots, f_{n-1})$ be the f-vector of a n-polytope P. Then,*

$$\begin{aligned} d(P) &\leq \lceil \log_2 f_0 \rceil, \\ d(P) &\leq \lceil \log_2(2f_1 n^{-1} - 1) \rceil + 1, \\ d(P) &\leq \lceil \log_2(f_2 - n + 2) \rceil + 2. \end{aligned}$$

One essential step in proving Theorem 2 is estimating how many of the considered faces are needed to cover all vertices. The depth bounds based on f_1 and f_2 rely on a vertex covering estimate that is not tight for all polytopes. Consequently, these depth bounds could still be improved.

An important remark is that Theorem 2 depends on ensuring that all faces of the same dimension have a common depth bound in order to apply Theorem 1. For $n \geq 4$, cyclic polytopes with an increasing number of vertices form a family with unbounded depth [1]. This implies that the idea behind Theorem 2 cannot be extended to obtain general bounds for $f_i, i \geq 4$, representing a notable limitation. However, the case of f_3 remains open, as no family of polyhedra with unbounded depth has been identified so far.

In this talk, we also present polytopes with increasing number of vertices and fixed depth, obtained from the Minkowski sum of cyclic polytopes and zonotopes, a relevant topic on this ongoing research.

References

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