An asymptotic rigidity property of chirotope extensions

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A *point configuration* in $\mathbb{R}^d$ is a labeled set of points $P = \{p_i\}_{i\in X}$ where X is a finite set, *the labels*, and $p_i \in \mathbb{R}^d$ for each $i \in X$. Formally, P is a map from X to $\mathbb{R}^d$, that is, an element of $(\mathbb{R}^d)^X$. The *orientation* of a $(d+1)$-tuple $(p_1, p_2, \dots, p_{d+1})$ of points in $\mathbb{R}^d$ is defined as

$$\chi(p_1, p_2, \dots, p_{d+1}) := \operatorname{sign}\det \begin{pmatrix} p_1 & p_2 & \dots & p_{d+1} \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

The *chirotope* of a point configuration $P \in (\mathbb{R}^d)^X$, with $|X| \geq d+1$, is the map

$$\chi_P\colon \begin{cases} X^{d+1} & \to \quad \{-,0,+\} \\ (i_1, i_2, \dots, i_{d+1}) & \mapsto \quad \chi\left(p_{i_1}, p_{i_2}, \dots, p_{i_{d+1}}\right) \end{cases}$$

sending each $(d+1)$-tuple of X to the orientation of the points they label. A *realizable chirotope* on a finite set X is a function $\omega : X^{d+1} \to \{-,0,+\}$, for some $d \geq 1$, that is the chirotope of some point configuration $P \in (\mathbb{R}^d)^X$, that is $\omega = \chi_P$; in that case, P is then a *realization* of ω. We say that a point configuration $P \in (\mathbb{R}^d)^X$ is *generic* if no $d+1$ points are contained in a common hyperplane, that is if χ_P takes values in $\{-,+\}$.

Let $X \subset Y$ be label sets. A point configuration $\widehat{P} \in (\mathbb{R}^d)^Y$ *extends* a point configuration $P \in (\mathbb{R}^d)^X$ if $\widehat{P}_{|X} := \{\widehat{p}_i\}_{i\in X}$ coincides with P. We say that an extension $\widehat{P}$ of P is a *generic extension* of P if no point $\hat{p} \in \widehat{P} \setminus P$ lies on a hyperplane spanned by points of $\widehat{P} \setminus \{\hat{p}\}$. (Note that this requirement is stronger than asking that $\widehat{P}_{|Y\setminus X}$ is generic but weaker than asking that $\widehat{P}$ is generic.) We say that a chirotope χ is *realizable on top of* a point configuration P if there exists an extension of P realizing χ.

Our main result is the following "rigidity" property:

Theorem 1 *Two full-dimensional point configurations P and Q in $\mathbb{R}^d$ are directly affinely equivalent if and only if for every finite generic extension $\widehat{P}$ of P, the chirotope of $\widehat{P}$ is realizable on top of Q.*

In coarser terms, we can say that two point configurations *have the same extensions* if every finite chirotope realizable on top of one is realizable on top of the other. Theorem 1 implies that for $d \geq 2$, two d-dimensional point configurations have the same extensions if and only if they are directly affinely equivalent. This is in sharp contrast with the situation for $d = 1$, as two point configurations in the real line have the same extensions if and only if their points are in the same order.

What happens if we bound from above the number of points added in the extension? Theorem 1 cannot generalize, not even for a single configuration P. Indeed, there are infinitely many convex quadrilaterals in $\mathbb{R}^2$ that are pairwise not affinely equivalent, whereas for every integer k there are only finitely many distinct sets of chirotopes on at most $4+k$ points.

Finite extensions nevertheless yield a quantitative analogue of Theorem 1, in which a single extension of bounded size can be used to discriminate all configurations that are "at least ε-far away" from a given configuration P:

Theorem 2 *For every $d \geq 2$ and every full-dimensional point configuration $P \in (\mathbb{R}^d)^X$ there exists constants $C(P)$ and $\tau(P) > 0$ such that for every $0 < \varepsilon \leq \tau(P)$, there exists a finite generic extension $\widehat{P}$ of P, with $\widehat{P} \setminus P$ of size at most $C(P)\log\frac{1}{\varepsilon}$ with the following property: for any configuration $Q \in (\mathbb{R}^d)^X$ such that $\chi_{\widehat{P}}$ can be realized on top of Q, there exists a direct affine transform φ such that $\max_{i\in X} \|p_i - \varphi(q_i)\|_2 \leq \varepsilon$.*

A preprint containing context and full proofs for these results is available at [1].

References

[1] Xavier Goaoc and Arnau Padrol. An asymptotic rigidity property from the realizability of chirotope extensions. Preprint, `arXiv:2505.14189`, 2025.

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