Shortest Descending Path is not solvable within ACM$\mathbb{Q}$

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Abstract

In this paper, we consider the Shortest Descending Path (SDP) Problem. The SDP Problem is the problem of computing a shortest path between two given points on a polyhedral terrain, with the additional constraint that the path may never increase in z coordinate. No polynomial algorithm is known to compute an SDP. In this paper, we present a construction demonstrating that an SDP is not computable within the ACM$\mathbb{Q}$ model, even in a terrain consisting of only three triangles. This result provides evidence of the difficulty of solving the problem exactly.

1 Introduction

A *polyhedral terrain* is a triangulated polyhedral surface such that every vertical line intersects it at a single point. Given a polyhedral terrain and two points s and t on the terrain, the *Shortest Descending Path* (SDP) problem consists of finding a shortest path from s to t such that the path is *descending* (i.e., its z-coordinate never increases as we move along it).

Descending paths on terrains were first studied by de Berg and van Kreveld [5], who proposed a data structure to determine if a descending path between two given points exists, in $O(\log n)$ time. The data structure uses linear space and can be built in $O(n \log n)$ time. The question of computing a shortest descending path was left open in [5]. Years later, Ahmed et al. [1] revisited the problem, proposing two approximation algorithms based on discretization. The exact computation of SDPs was first explored by Ahmed and Lubiw [2], who showed that the main difficulty of the problem lies on finding the shortest descending path that goes through a given sequence of triangles. Indeed, they show that the SDP problem can be reduced to finding an SDP through a given triangle sequence.

Exact algorithms have been proposed for some particular cases, such as convex [9], pseudo-convex, and pseudo-orthogonal terrains [2]. Various $(1+\varepsilon)-$approximation algorithms have been proposed by different authors in recent years. For instance, Ahmed et al. [1] considered using Steiner points, a technique that involves placing additional points instead of directly connecting edges, which are then utilized by a path finding algorithm. Cheng and Jin [4] follow a different approach, based on methods for shortest paths on polyhedra.

In this work, we show that there are intrinsic algebraic difficulties in computing SDPs. We do this by showing that SDPs cannot be computed in the *Algebraic Computation Model over the Rational Numbers* (ACM$\mathbb{Q}$). This model was first used in computer science by Bajaj [3][1]. In the ACM$\mathbb{Q}$, one can exactly compute any number obtained from rational numbers by applying operations $+, -, \times, \div$ and $\sqrt[k]{}$, for any integer $k \geq 2$. More recently, De Carufel et al. [7] applied the model to show that the *weighted region problem* (WRP) cannot be solved within the ACM$\mathbb{Q}$ model (even if we restrict the problem to a single region it is still unsolvable within ACM$\mathbb{Q}$, see [6]). The WRP consists in finding a shortest path between two points in a weighted subdivision, where the length of each portion of the path inside a region gets multiplied by the region weight. Interestingly, Ahmed and Lubiw [2] observed that the SDP problem is similar to the WRP, in the sense that the bend angles of SDPs are related in a way similar to bend angles of solutions to the WRP, concluding that the SDP problem is as hard, or even harder, than the WRP.

In this work, we take the similarities between the WRP and the SDP (which are particular cases of the *Shortest Anisotropic Path* problem[8]) problem one step further, by showing that, as with the WRP, the SDP problem cannot be solved exactly within the ACM$\mathbb{Q}$. This gives new evidence of the unsolvability of the problem.

2 Preliminaries

2.1 Shortest Descending Path

A *polyhedral terrain*, or just *terrain*, is given as a triangulation of a set of points in the plane, where each point has an elevation. It can be also considered as an xy-monotone polyhedral surface.

[1]The name ACM$\mathbb{Q}$ was coined by De Carufel et al. [7].

A *geodesic path* is, locally, the shortest path on a surface between two points.

Given two points on the terrain s and t, where the elevation of t is not higher than that of s, we consider *descending geodesic paths* from s to t as geodesic paths where the z-coordinate along the path from s to t never increases. Moreover, if such a path is the shortest one among all possible descending geodesic paths, we call it a *shortest descending path*.

Ahmed and Lubiw [2] showed that, given a sequence of triangles, there is at most one shortest descending path. They also demonstrated that the main challenge in the SDP problem is computing the exact SDP through a given triangle sequence.

2.2 Galois Theory

Galois Theory is a branch of abstract algebra that, at its core, answers the question of which polynomials can be solved using radicals—operations involving addition $(+)$, subtraction $(-)$, multiplication $(\cdot)$, division $(\div)$, and root extraction $(\sqrt[n]{\cdot})$. This is relevant, as the ACM$\mathbb{Q}$ is restricted to work with numbers that can be represented as radical expressions, meaning they are obtained through a finite sequence of these operations.

De Carufel et al. [7] presented a criterion to determine when a polynomial is solvable in the ACM$\mathbb{Q}$ model.

Lemma 2.1 *(See [7]) Let $p(x)$ be a polynomial of even degree $d \geq 6$. Suppose that there are three prime numbers q_1, q_2 and q_3 that do not divide the discriminant $\Delta(p(x))$ of $p(x)$ such that*

$$\begin{aligned} p(x) &\equiv p_d(x) && \mod q_1 \\ p(x) &\equiv p_1(x)p_{d-1}(x) && \mod q_2 \\ p(x) &\equiv p_1'(x)p_2(x)p_{d-3}(x) && \mod q_3 \end{aligned}$$

where $p_i(x)$ denotes an irreducible polynomial of degree i modulo q_j. Then $p(x) = 0$ is unsolvable within the ACM$\mathbb{Q}$ model.

3 Construction

Our proof consists on finding a particular construction, an instance of the SDP problem composed of a small set of triangles, that represents a terrain and two points on it, such that computing the SDP from one point to the other involves solving $p(x) = 0$ for a polynomial that satisfies the conditions of unsolvability of Lemma 2.1.

Next we describe our construction. Consider the problem of finding the Shortest Descending Path from $A = (-4, 10, 6)$ to $D = (4, 2, -4)$ embedded in the mesh defined by the following points:

$A = (-4, 10, 6)$
$O = (0, 0, 0)$
$B = (1, 8, 1)$
$C = (4, 4, 2)$
$D = (4, 2, -4)$

The terrain is defined by only three triangles: $\Delta OAB, \Delta OBC$ and ΔOCD (Figure 1).

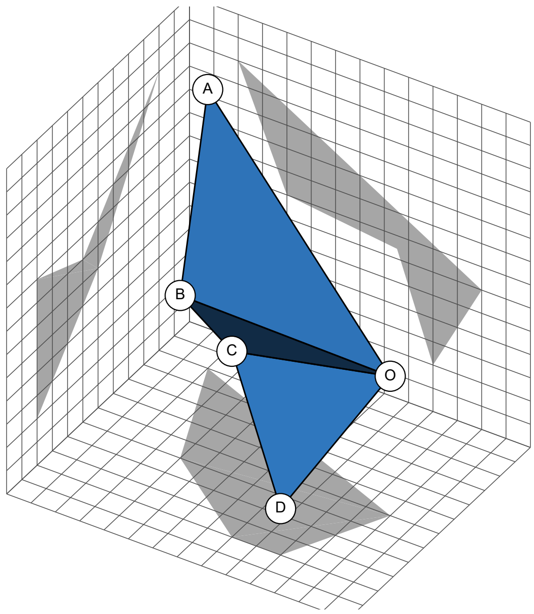

Figure 1: A visual representation of the instance, in 3D. The shadows are parallel projections to help visualize the 3D structure.

We define two parameters $t_1, t_2 \in [0, 1]$ and we define two points $P_1 = Bt_1 + O(1 - t_1) = Bt_1$ and $P_2 = Ct_2 + O(1 - t_2) = Ct_2$. Any path from A to D can be expressed as a sequence $A \to P_1 \to P_2 \to D$, for two values t_1, t_2. Consequently, the distance from A to D, parametrized by t_1 and t_2 (Figure 2), is given by:

$$\begin{aligned} d(t_1, t_2) = &\sqrt{(2 - 4t_2)^2 + (4 - 4t_2)^2 + (-2t_2 - 4)^2} + \\ &\sqrt{(-8t_1 + 4t_2)^2 + (-t_1 + 2t_2)^2 + (-t_1 + 4t_2)^2} + \\ &\sqrt{(t_1 - 6)^2 + (t_1 + 4)^2 + (8t_1 - 10)^2} \end{aligned}$$

Because the optimal solution lays precisely under the boundary condition enforced by the non-ascending constraint (Figure 2), we know that $t_2 = \frac{B.z}{C.z} t_1 = \frac{1}{2} t_1$, and thus we can write the distance solely in terms of t_1:

$$\begin{aligned} d(t_1) = &\sqrt{(2 - 2t_1)^2 + (4 - 2t_1)^2 + (-t_1 - 4)^2} + \\ &\sqrt{37}\sqrt{t_1^2} + \\ &\sqrt{(t_1 - 6)^2 + (t_1 + 4)^2 + (8t_1 - 10)^2} \end{aligned}$$

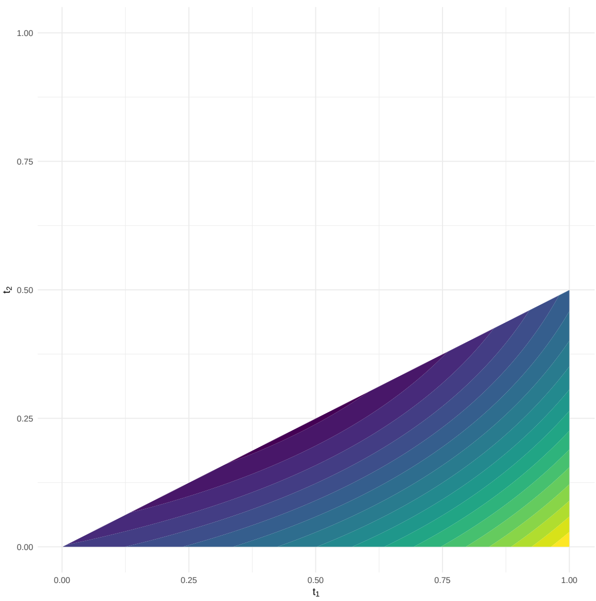

Figure 2: Contour Plot of the distance function with two parameters t_1 and t_2. Darker is lower, with 17.7 to 17.9 the darkest and 21.3 to 21.5 the lightest. The empty region is the region of points that would make the path non-descending, therefore the diagonal of the boundary represents the no-ascending constraint. We can see that for this instance the best solutions are found near this constraint.

For $0 < t_1 \leq 1$, this function is differentiable, and thus we can consider its derivative:

$$\frac{d}{dt_1}d(t_1) = \frac{9t_1 - 8}{\sqrt{(2-2t_1)^2 + (4-2t_1)^2 + (-t_1-4)^2}} + \frac{\sqrt{37}\sqrt{t_1^2}}{t_1} + \frac{66t_1 - 82}{\sqrt{(t_1-6)^2 + (t_1+4)^2 + (8t_1-10)^2}}$$

We write it all under the same denominator and set it equal to 0:

$$\frac{d}{dt_1}d(t_1) = \frac{33\sqrt{2}t_1^2\sqrt{9t_1^2-16t_1+36}}{t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}} + \frac{9t_1^2\sqrt{33t_1^2-82t_1+76}}{t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}} - \frac{41\sqrt{2}t_1\sqrt{9t_1^2-16t_1+36}}{t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}} - \frac{8t_1\sqrt{33t_1^2-82t_1+76}}{t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}} + \frac{\sqrt{37}\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}\sqrt{t_1^2}}{t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}} = 0$$

The denominator is equal to 0 if and only if $t_1 = 0$, so we can multiply both sides by the denominator.

$$\begin{aligned} 33\sqrt{2}t_1^2\sqrt{9t_1^2-16t_1+36} &+ \\ 9t_1^2\sqrt{33t_1^2-82t_1+76} &- \\ 41\sqrt{2}t_1\sqrt{9t_1^2-16t_1+36} &- \\ 8t_1\sqrt{33t_1^2-82t_1+76} &+ \\ \sqrt{37}\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76}\sqrt{t_1^2} &= 0 \end{aligned}$$

Because $t_1 \geq 0$, we have $\sqrt{t_1^2} = t_1$

$$\begin{aligned} 33\sqrt{2}t_1^2\sqrt{9t_1^2-16t_1+36} &+ \\ 9t_1^2\sqrt{33t_1^2-82t_1+76} &- \\ 41\sqrt{2}t_1\sqrt{9t_1^2-16t_1+36} &- \\ 8t_1\sqrt{33t_1^2-82t_1+76} &+ \\ \sqrt{37}t_1\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76} &= 0 \end{aligned}$$

Since we have $0 < t_1$, we can divide by t_1.

$$\begin{aligned} 33\sqrt{2}t_1\sqrt{9t_1^2-16t_1+36} &+ \\ 9t_1\sqrt{33t_1^2-82t_1+76} &- \\ 41\sqrt{2}\sqrt{9t_1^2-16t_1+36} &- \\ 8\sqrt{33t_1^2-82t_1+76} &+ \\ \sqrt{37}\sqrt{9t_1^2-16t_1+36}\sqrt{33t_1^2-82t_1+76} &= 0 \end{aligned}$$

Some algebraic manipulations lead us to the fact that the only root in $[0, 1]$ of this function is the smallest real root of the following polynomial:

$$20552697x^8 - 175216932x^7 + 738557316x^6 - 1914201240x^5 + 3347241359x^4 - 3951992296x^3 + 3144512520x^2 - 1805138080x + 436621424$$

Next we show that this polynomial is not solvable by radicals. To that end, we need three primes q_1, q_2, q_3 that adhere to the conditions of Lemma 2.1. In particular, we find that the polynomial is equivalent to:

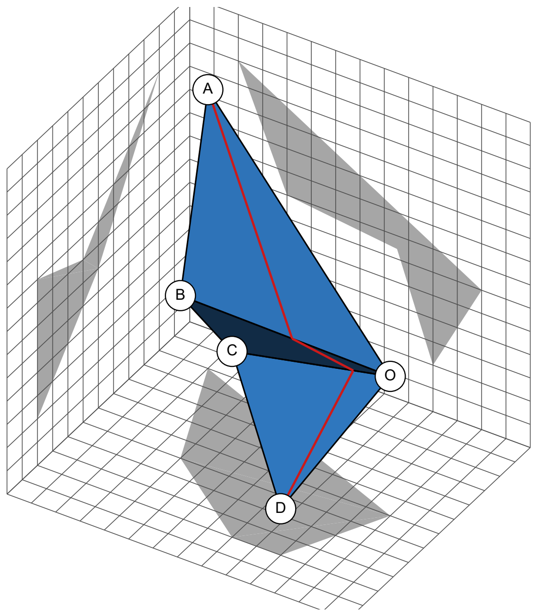

Figure 3: The red path represents the Shortest Descending Path on this problem. Note that the line is less straight than it could be just to enforce the no-ascending constraint.

$$\begin{aligned} & 45(x^8 + 49x^7 + 38x^6 + \\ & 48x^5 + 51x^4 + 5x^3 + 4x^2 + 53x + 1) \quad \bmod 67 \\ & 3(x + 15)(x^7 + 6x^4 + \\ & 15x^3 + 7x^2 + x + 8) \quad \bmod 17 \\ & 17(x + 3)(x^2 + 9x + 17)(x^5 + 5x^4 + \\ & 13x^3 + 7x^2 + 12x + 11) \quad \bmod 19 \end{aligned}$$

Therefore, we can take $q_1 = 67, q_2 = 17, q_3 = 19$, and as a consequence we conclude that even for such a simple example with only 3 triangles, the Shortest Descending Path problem is not solvable within the ACM$\mathbb{Q}$.

Theorem 3.1 *The shortest descending path problem cannot be solved exactly within the ACM$\mathbb{Q}$.*

Figure 3 shows a numeric approximation of the SDP, obtained for $t_1 \approx 0.47$ and $t_2 \approx 0.23$.

4 Conclusions

We analyzed the SDP problem and showed that even for very simple constructions with only three triangles (in fact, the problem remains difficult with at least three triangles), it is impossible to obtain an exact formula under ACM$\mathbb{Q}$. While the general computational complexity of the Shortest Descending Path problem remains unknown, our results suggest that determining its complexity may be related to algebraic complexity. Future research could explore this problem under a computational model over the reals. In addition, we may want to analyze the effectiveness of approximation algorithms under ACM$\mathbb{Q}$, and identify other families of polyhedral terrains for which exact algorithms exist.

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