Metric representation of graphs

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The *metric representation* of a vertex u in a connected graph G with respect to an ordered vertex subset $W = \{\omega_1, \dots, \omega_n\} \subseteq V(G)$ is the vector $r(u|W) = (d(x,\omega_1), \dots, d(x,\omega_n))$ of distances from u to the vertices of W. The subset W is a *resolving set* of G if $r(u|W) \neq r(v|W)$, for every $u, v \in V(G)$ with $u \neq v$. Thus, a resolving set with n elements provides a set S of metric representation vectors, where $S \subset \mathbb{Z}^n$ and its cardinality is equal to the order of the graph.

Resolving sets are used to distinguish the vertices of a graph and have become a topic of much interest in the graph theory community due to its applications in diverse areas.

In this work, we are interested in the reverse problem, that is, given a finite subset $S \subset \mathbb{Z}^n$, determining whether there exist a graph G and a resolving set W of G such that S is the set of metric representations of the vertices of G with respect to W. In such a case, we say that S is *realizable* and (G, W) is a *realization* of S.

For example, different realizations of the set $S = \{(0,3), (3,0), (1,2), (2,1), (2,3), (3,2), (1,4), (4,1), (2,5), (5,2), (3,4), (4,3), (4,5), (5,4)\}$ are depicted in Figure 1, where black vertices are those in W.

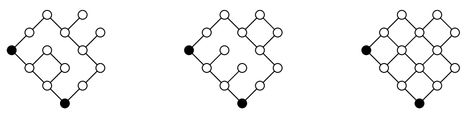

Figure 1: Three realizations of the set S.

Our main goal consists of characterizing realizable sets S of $\mathbb{Z}^n$ and determining which properties of the graphs realizing a subset S can be derived from the set S of metric coordinates.

The strong product of paths plays an important role in our approach, since, roughly speaking, every graph realizing a set $S \subseteq \mathbb{Z}^n$ can be viewed as a subgraph of a strong product of n paths by identifying the vertex u with the point $r(u|W)$ of an n-dimensional grid.

Some questions were already solved in [1]. Concretely, realizable sets were characterized and the uniqueness of the realization was discussed. The main result is the following, where x_i denotes the i-th coordinate of any $x \in \mathbb{Z}^n$.

Theorem 1 *[1] A subset $S \subset \mathbb{Z}^n$ is realizable if and only if the following properties hold.*

i) If $x \in S$, then $x_i \geq 0$ for every $i \in [n]$. Moreover, x has at most one coordinate equal to zero.

ii) For every $i \in [n]$, there exists exactly one element $x \in S$ such that $x_i = 0$.

iii) If $x \in S$ and $x_i > 0$ for $i \in [n]$, then there exists $y \in S$ satisfying $y_i = x_i - 1$ and $\max_{j\in[n]}\{|y_j - x_j|\} \leq 1$.

Recently, we have characterized the sets $S \subset \mathbb{Z}^n$ that can be realized by a tree. In such a case, there is only one tree that realizes S and we have also characterized when this tree is the only realization of S.

Theorem 2 *If $S \subseteq \mathbb{Z}^n$ is a realizable set, then there exists a realization (T, W) of S such that T is a tree if and only if the following conditions hold*

i) for every $x, y \in S$, if $\max_{i\in[n]} |x_i - y_i| = 1$, then $y_j \neq x_j$ for every $j \in [n]$;

ii) for every $x \in S_0$ and every $j \in [n]$ such that $x_j > 0$, there exists exactly one element $y \in S_0$ such that $\max_{i\in[n]} |x_i - y_i| = 1$ and $y_j = x_j - 1$.

Proposition 3 *If $S \subseteq \mathbb{Z}^n$ is realizable by a tree T, then only T realizes S if and only if $\max_{i\in[n]} |x_i - y_i| > 1$ for every pair of different vertices $x, y \in S_0^*$, where $S_0^* = \{x \in S_0 : (x_1 + 1, \dots, x_n + 1) \in S\}$.*

In future work, we want to analyse the structure and more properties of graphs that can be described from the set of metric coordinates.

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References

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