Chromatic quasisymmetric functions for signed graphs

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1 Classical graph coloring

A proper coloring of a simple graph $G = (V, E)$ is a map κ from the vertex set $V = \{v_1, ..., v_d\}$ to the positive integers $\mathbb{N} = \{1, 2, ...\}$ such that adjacent vertices are assigned different colors. Consider the set of commuting variables $x = (x_1, x_2, ...)$ *the chromatic symmetric function of* G is defined as

$$X_G(x) := \sum_{\kappa \text{ proper}} x^\kappa = \sum_{\kappa \text{ proper}} x_{\kappa(1)} \cdots x_{\kappa(v_d)}. \quad (1)$$

It has generated extensive research since its introduction in [3]. It is clear that $X_G \in Sym$, where Sym is the algebra of symmetric functions. A remarkable feature of this function is that it specializes in the chromatic polynomial $\chi_G(k)$.

In [1] Ellzey introduced a refinement of Stanley's chromatic symmetric function for directed graphs. Given a simple directed graph $\overrightarrow{G} = (V, E)$ the *chromatic quasisymmetric function of* $\overrightarrow{G}$ is

$$X_{\overrightarrow{G}}(x, t) := \sum_{\kappa \text{ proper}} t^{asc(\kappa)} x^\kappa \quad (2)$$

where $asc(\kappa) := \#\{(u, v) \in E : \ \kappa(u) < \kappa(v)\}$. It is straightforward that Stanley's invariant is recovered by evaluating this function in $t = 1$. Also, it is easy to check that $X_{\overrightarrow{G}}(x, t)$ lies in $QSym[t]$ where $QSym$ is the algebra of quasisymmetric functions. The importance of the algebra $QSym$ in algebraic combinatorics is well-known.

2 Signed graph coloring

A *signed graph* is a simple graph in which each edge has either a + or a − associated with it, then we call this edges positive edges or negative edges respectively. In [4] Zaslavsky defined what a *proper coloring of a signed graph* is. Given a signed graph Σ with vertex set V, a proper coloring of Σ is a map $\kappa : V \to \mathbb{Z}$ verifying $\kappa(u) \neq \epsilon\kappa(v)$ if $\{u, v\}$ is an edge of Σ and ϵ is its sign.

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An analog of Stanley's chromatic symmetric function in the framework of signed graph coloring was introduced in [2]. Given a signed graph on d vertices Σ, consider the set of commuting variables $\mathbf{x} := (..., x_{-1}, x_0, x_1, ...)$, then *the chromatic signed symmetric function of* Σ is

$$X_\Sigma(\mathbf{x}) := \sum_{\kappa \text{ proper}} \mathbf{x}^\kappa = \sum_{\kappa \text{ proper}} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_d)}.$$

This function is invariant under the action of the signed symmetric group $S\mathfrak{S}$ on its variables, in other words, it belongs to $SSym$, the algebra of signed symmetric functions. Zaslavsky also developed a theory for the orientation of signed graphs and proves that there exists a bijection between the acyclic orientations of a signed graph on d vertices Σ and the connected components of the complement of an hyperplane arrangement $H_\Sigma \in \mathbb{R}^d$. This beautiful result generalizes the well-known analog result for the classical case.

The aim of this paper is, taking account of this bijection define an invariant $X_{\overrightarrow{\Sigma}}(\mathbf{x}, t)$ for signed directed graphs $\overrightarrow{\Sigma}$ that refines $X_\Sigma(\mathbf{x})$ in the same way as $X_{\overrightarrow{G}}(x, t)$ refined $X_G(x)$. In other words, we want to fill the gap in the following table and to study the naturally arising algebra.

	undirected	directed
G	$X_G(x) \in Sym$	$X_{\overrightarrow{G}}(x, t) \in QSym[t]$
Σ	$X_\Sigma(\mathbf{x}) \in SSym$	?

We define bases for this algebra, propose a formula for calculating the invariant and present some symmetry results.

References

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