Ordered Yao graphs: maximum degree, edge numbers, and clique numbers

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Abstract

For a positive integer k and an ordered set of n points in the plane, define its *k-sector ordered Yao graphs* as follows. Divide the plane around each point into k equal sectors and draw an edge from each point to its closest predecessor in each of the k sectors. We analyze several natural parameters of these graphs. Our main results are as follows:

I Let $d_k(n)$ be the maximum integer so that for every n-element point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph has maximum degree at least $d_k(n)$. We show that $d_k(n) = n - 1$ if $k = 4$ or $k \geq 6$, and provide some estimates for the remaining values of k. Namely, we show that $d_1(n) = \Theta(\log_2 n)$; $\frac{1}{2}(n-1) \leq d_3(n) \leq 5\left\lceil\frac{n}{6}\right\rceil - 1$; $\frac{2}{3}(n-1) \leq d_5(n) \leq n-1$;

II Let $e_k(n)$ be the minimum integer so that for every n-element point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph has at most $e_k(n)$ edges. Then $e_k(n) = \left\lceil\frac{k}{2}\right\rceil \cdot n - o(n)$.

III Let w_k be the minimum integer so that for every point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph has clique number at most w_k. Then $\lceil\frac{k}{2}\rceil \leq w_k \leq \lceil\frac{k}{2}\rceil + 1$.

All the orders mentioned above can be constructed effectively.

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1 Introduction and main results

For a point set in the plane, define its *Yao graphs* in the following way. Fix an integer $k \geq 1$, and divide the plane around each point into k equal sectors such that one boundary ray is horizontal and directed to the right. Then draw an edge from each point to its closest neighbor in each of the k sectors, see Figure 1, top. Let us call the resulting directed graph a *k-sector Yao graph*. Observe that the outdegree of every vertex is at most k, and is strictly smaller if some of the corresponding sectors are empty, as in Figure 1, top.

The notion of an *ordered Yao graph* is closely related. In this case, the vertices appear one by one, and each new vertex has precisely one outgoing edge towards its closest predecessor, see Figure 1, bottom. To make this notion well-defined, we also need to specify how to break the ties when a point lies on a sector-bounding ray or if two points are at the same distance from a third one and within the same sector of it. One way for the former is to regard each sector-bounding ray as a part of the next sector, say, in a counterclockwise direction; for the latter, it is natural to order the points with the same distance from p and within the same sector of p in, say, a counterclockwise order around p (so the one on the sector boundary is the 'closest'). To simplify the arguments, all point sets considered are *in general position*, i.e., neither of the scenarios described above occurs. Also, whenever defining an ordering that applies for all point sets, it is legitimate to assume that the first condition is true, otherwise we may rotate the set by a sufficiently small angle.

Yao graphs were introduced by Yao [22], while their ordered variants are due to Bose, Gudmundsson, and Morin [5] (defined for the slightly different variant called theta graphs). In modern Computational Geometry, these graphs are used to construct geometric spanners with nice additional properties, such as logarithmic maximum degree and logarithmic diameter. In [7], Bose et al. showed that the *stretch factor* of k-sector Yao graphs is at most $1/(\cos(2\pi/k)-\sin(2\pi/k))$

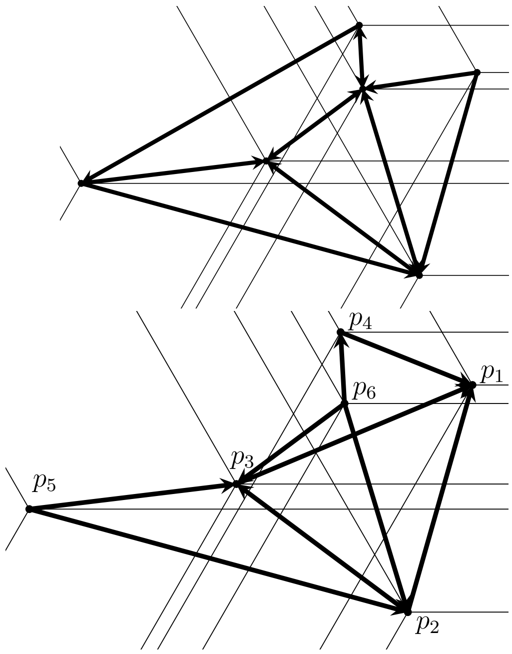

Figure 1: Unordered (t) and ordered (b) 3-sector Yao graphs on the same set of six points.

for $k \geq 9$, which has been improved several times later, for example in [3]. In [5], the authors also study the stretch factor for the ordered variant. Sparse graphs with a small dilation have also been studied in [4, 6, 7], among others.

Note that Yao graphs in the special case $k = 1$ are the well-known Nearest Neighbor Graphs with numerous applications in Computational Geometry such as computing geometric shortest paths, spanners, well-separated pairs, and approximate minimum spanning trees in, see the survey [21], books [12, 19], or monograph [18]. The ordered variant of the Nearest Neighbor Graphs was introduced in [1, 13] in the context of dynamic algorithms.

A systematic study of the basic combinatorial properties of Nearest Neighbor Graphs dates back at least to the classical paper [14] by Eppstein, Paterson, and Yao in which, among other, they made the following simple observation: two edges with the same endpoint meet at an angle of at least $\pi/3$, and thus the maximum indegree is bounded from above by 6 for planar point sets. However, it is not hard to see that for any $k > 1$, the maximum indegree of a k-sector Yao graph can be arbitrarily large. It is an open problem in the field to determine whether every point set P admits an order such that the maximum indegree of the corresponding ordered Yao graph is bounded from above by some constant c_k, independent of the size of P, see [5]. To get a better understanding of how degrees behave in these graphs, here we attempt to *maximize* the maximum indegree. We addressed the special case $k = 1$ in a separate paper [2].

Definition 1 *For $k, n \in \mathbb{N}$, let $d_k(n)$ be the maximum indegree one can always guarantee in an n-vertex k-sector ordered Yao graph by picking a suitable order. In other words, $d_k(n)$ is the maximum integer satisfying the following property. For every n-point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph has maximum indegree at least $d_k(n)$.*

Theorem 2 *The following bounds hold:*

1. $d_1(n) = \Theta(\log n)$;
2. $\frac{1}{2}(n-1) \leq d_3(n) \leq 5\left\lceil \frac{n}{6} \right\rceil - 1$;
3. $\frac{2}{3}(n-1) \leq d_5(n) \leq n-1$;
4. $d_k(n) = n-1$ *otherwise, that is, if* $k = 2, 4$ *or* $k \geq 6$.

Next we study the number of edges of a k-sector ordered Yao graph. This is trivial for $k = 1$. Indeed, for every n-point set P and every order of it, all the vertices of the corresponding ordered Yao graph, but the first one, have precisely one outgoing edge, and thus the graph always contains $n-1$ edges. Assume therefore that $k \geq 2$. Our next result determines the maximum number of edges of a k-sector ordered Yao graph that we *can always guarantee* by picking a suitable order. We also obtain a 'complementary' result regarding the maximum number of edges that we *sometimes cannot avoid* regardless of the order we take.

Definition 3 *For $k, n \in \mathbb{N}$, let $E_k(n)$ be the maximum number of edges one can always guarantee in an n-vertex k-sector ordered Yao graph by picking a suitable order. In other words, $E_k(n)$ is the maximum integer satisfying the following property. For every n-point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph contains at least $E_k(n)$ edges.*

Theorem 4 *For $k \neq 3$, we have $E_k(n) = 2n - 3$ for $n \geq 3$, and $E_3(n) = 2n - 4$ for $n \geq 4$.*

Definition 5 *For $k, n \in \mathbb{N}$, let $e_k(n)$ be the maximum number of edges one sometimes cannot avoid in an n-vertex k-sector ordered Yao graph regardless of the picked order. In other words, $e_k(n)$ is the maximum integer satisfying the following property. There exists an n-point set in the plane such that for every order, the corresponding k-sector ordered Yao graph contains at least $e_k(n)$ edges.*

Theorem 6 *For each fixed $k \geq 2$, we have $e_k(n) = n \cdot \left\lceil \frac{k}{2} \right\rceil - o(n)$ as $n \to \infty$. Moreover, if $k \geq 4$, then $n \cdot \left\lceil \frac{k}{2} \right\rceil - O\left(k^3 \cdot \sqrt{n}\right) \leq e_k(n) \leq n \cdot \left\lceil \frac{k}{2} \right\rceil - \Omega\left(k \cdot \sqrt{n}\right)$.*

Finally, we study the *clique number* of the the k-sector ordered Yao graph, defined as the size of its largest subset of pairwise adjacent vertices, where we omit the orientation of the edges. As before, there is nothing to study if $k = 1$. Indeed, all the vertices but the first one have precisely one outgoing edge, and thus the Yao graph is triangle-free, and in fact, it is not hard to show that the Yao graph is always *acyclic*. So we can assume that $k \geq 2$. The following two 'complementary' results determine the maximum size of a clique that we can always achieve by taking a suitable order of the point set, and estimate the maximum size of a clique that we sometimes cannot avoid regardless of the order we take.

Definition 7 *For $k, n \in \mathbb{N}$, let $W_k(n)$ be the maximum clique number one can always guarantee in an n-vertex k-sector ordered Yao graph by picking a suitable order. In other words, $W_k(n)$ is the maximum integer satisfying the following property. For every n-point set in the plane, there exists an order such that the corresponding k-sector ordered Yao graph contains a clique of size $W_k(n)$.*

Theorem 8 *For $k \geq 2$ and $n \geq 3$ we have $W_k(n) = 3$ with the only exception being $W_3(3) = 2$.*

Definition 9 *For $k, n \in \mathbb{N}$, let $w_k(n)$ be the maximum clique number one sometimes cannot avoid in an n-vertex k-sector ordered Yao graph regardless of the picked order, and $w_k = \sup_n w_k(n)$. In other words, w_k is the maximum integer satisfying the following property. There exists point set in the plane such that for every order, the corresponding k-sector ordered Yao graph contains a clique of size w_k.*

Theorem 10 *For $k \geq 2$ we have $\lceil \frac{k}{2} \rceil \leq w_k \leq \lceil \frac{k}{2} \rceil + 1$.*

It is not hard to see that, as a function of n, $w_k(n)$ is monotonically non-decreasing and bounded from above by $k + 1$, the maximum outdegree increased by 1. Hence, $w_k(n) = w_k$ for all sufficiently large n.

Remark. Due to the page limit, all the proofs of the aforementioned results are presented in the Appendix.

Related work. In this paper, we define closest neighbors based on the Euclidean distance. However, there are alternative ways. For instance, one may want to minimize the distance between a point and the orthogonal projection of its neighbor on the bounding ray of the hosting sector (or, in another variant, on the bisector of the hosting sector). The resulting graphs, usually referred to in the literature as *θ-graphs*, were introduced by Clarkson [9] and independently by Keil [17], while their ordered variants are due to Bose, Gudmundsson, and Morin [5]. The spanning ratio of these graphs is at most $1/(1 - 2\sin(\pi/k))$ for $k \geq 7$ and any order of the point set [20], see also [4, 10, 11].

2 Concluding remarks

Our Theorems 2, 6 and 10 leave room for improvements. For the first, it would be interesting to find out whether our simple bounds $\frac{1}{2}(n-1) \leq d_3(n)$ and $d_5(n) \leq n - 1$ are tight or not.

In the setting of Theorem 10, we suspect that the upper bound $w_k \leq \lceil \frac{k}{2} \rceil + 1$ may be tight since we confirmed it for $k = 3, 4$. However, we also showed that $w_k \left(\lceil \frac{k}{2} \rceil + 1 \right) = \lceil \frac{k}{2} \rceil$ for $k \geq 4$. In other words, a point set for which a clique of size $\lceil \frac{k}{2} \rceil + 1$ is unavoidable contains more than $\lceil \frac{k}{2} \rceil + 1$ points.

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A Notation

We assume that one of the sector-bounding rays of the k-sector Yao graph is a horizontal ray directed to the right and call it ℓ_0, and the remaining rays are $\ell_1, \ell_2, ..., \ell_{k-1}$ in a counterclockwise direction from ℓ_0. Denote the sectors between these rays by $s_0, s_1, ..., s_{k-1}$ in counterclockwise order. For instance, s_0 is the sector bounded by ℓ_0 and ℓ_1, and so forth. Let us call the antipodal rays $-\ell_0, \ldots, -\ell_{k-1}$ the *dual rays*. Drawn from a common origin, they separate the plane into k *dual sectors* labeled by $-s_0, ..., -s_k$. It is easy to check that p belongs to the ith sector of q if and only if q belongs to the ith dual sector of p, see Figure 2.

The notation AnyOrder(Q) stands for an arbitrary order of the point set Q.

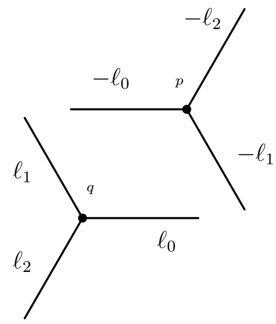

Figure 2: p lies in the first sector of q if and only if q lies in the first dual sector of p. Or to be short, $p \in s_0(q)$ and $q \in -s_0(q)$.

B Maximum degree

Let us briefly outline the strategy of the proof. First, note that the equality $d_1(n) = \Theta(\log n)$ is immediate from Theorem 1 and Theorem 2 in [2]. So we subsequently assume that $k > 1$. In Section B.1, we present a simple order yielding that $d_k(n) = n - 1$ for all even k, and that $d_k(n) \geq \frac{1}{2}(n-1)$ for all odd $k \geq 3$. In Section B.2, we give another simple order yielding that $d_k(n) = n - 1$ for all $k \geq 6$. A combination of these two orders implies that $d_5(n) \geq \frac{2}{3}(n-1)$. Finally, in Section B.3, we prove that $d_3(n) \leq 5\left\lceil \frac{n}{6} \right\rceil - 1$ via an explicit construction. Together these results complete the proof of Theorem 2.

B.1 Orthogonal enumeration

Lemma 11 *Let Q be a point set contained in $t = \lfloor \frac{k}{2} \rfloor$ cyclically consecutive dual sectors of a point p. Then there exists an ordering such that the corresponding k-sector Yao graph contains all the directed edges $q \to p$ for $q \in Q$.*

Proof. Due to the symmetry of this statement, it is sufficient to prove it only for the points from dual sectors between $-\ell_0$ and $-\ell_t$. Let $q_1, \ldots, q_m$ be these points labeled in nondecreasing order of their y-coordinates. We claim that in the k-sector Yao graph corresponding to the ordering

$$p, q_1, \ldots, q_m, \text{AnyOrder}(P \setminus \{p, q_1, \ldots, q_m\}),$$

each q_i is adjacent to p. Indeed, all the previous $q_j, j < i$, lie below q_i by construction. At the same time, p belongs to one of the first t sectors of q_i, and thus p lies above q_i since $t \le \frac{k}{2}$. So p is the unique point in one of the first t sectors of q_i, and thus $q_i \to p$ is an edge. □

Proposition 1 *For all $n \in \mathbb{N}$, we have $d_k(n) = n - 1$ if k is even, and $d_k(n) \ge \frac{1}{2}(n-1)$ if $k > 1$ is odd.*

Proof. Let p be the highest point of P. Note that $t = \lfloor \frac{k}{2} \rfloor$ of its dual sectors that lie in the upper half-plane are empty by construction.

If k is even, then the remaining $n - 1$ points are distributed between t dual sectors in the lower half-plane, and Lemma 11 implies that all of them can be adjacent to p under a suitable ordering. Therefore, $d_k(n) = n - 1$, as desired.

If k is odd, the remaining $n-1$ points are distributed between $t + 1$ dual sectors intersecting the lower half-plane. By the pigeonhole principle, either the first t or the last t contain at least half of the points. Moreover, all of them can be adjacent to p under a suitable ordering according to Lemma 11. Therefore, $d_k(n) \ge \frac{1}{2}(n-1)$, as desired. □

B.2 Radial enumeration

Proposition 2 *For all $n \in \mathbb{N}$, $k \ge 6$, we have $d_k(n) = n - 1$.*

Proof. Let P be any point set and p be an arbitrary point of P. Label the remaining points $q_1, \ldots, q_{n-1}$ such that their Euclidean distances to p are nonincreasing. We claim that the indegree of p in the k-sector Yao graph under the ordering $p, q_1, \ldots, q_{n-1}$ equals $n - 1$. Assume the contrary, namely that for some $j < i$, q_i is adjacent to q_j instead of p. This implies that p and q_j belong to the same k-sector of q_i, and thus $\measuredangle pq_iq_j < 2\pi/k \le \pi/3$. Therefore, $\angle pq_iq_j$ is not a largest angle of the triangle q_ipq_j and so pq_j is not a longest side. Since $|q_ip| \le |pq_j|$ by construction, we conclude that q_iq_j is a longest side of the triangle. However, in this case q_i should be adjacent to p instead of q_j, a contradiction. □

Proposition 3 *For all n, we have $d_5(n) \ge \frac{2}{3}(n-1)$.*

Proof. Let P be any point set and p be the highest point of P, so for $k = 5$, $-s_3(p)$ and $-s_4(p)$ are empty. Let P_0, P_1, P_2 be sets of the remaining points in $-s_0(p)$, $-s_1(p)$ and $-s_2(p)$, respectively, and a_0, a_1, a_2 be their cardinalities, see Figure 3.

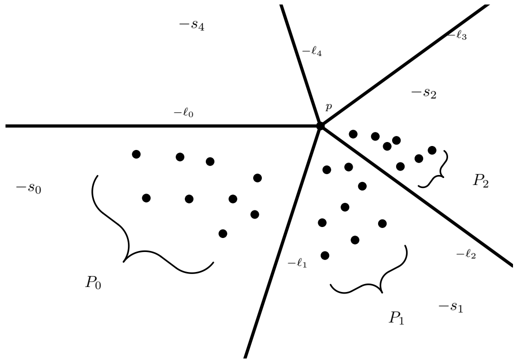

Figure 3: Five dual sectors of p; the first two are empty.

Lemma 11 implies that there exists an ordering such that each vertex of $P_0 \cup P_1$ is adjacent to p. Besides, there exists an ordering such that each vertex of $P_1 \cup P_2$ is adjacent to p. A similar statement for the union $P_0 \cup P_2$ will complete the proof. Indeed, since $a_0 + a_1 + a_2 = n - 1$, at least one of the sums $a_0 + a_1, a_1 + a_2, a_2 + a_0$ is at least $\frac{2}{3}(n-1)$ by the pigeonhole principle.

Let $q_1, \ldots, q_m$ be the points of $P_0 \cup P_2$ labeled such that their (Euclidean) distances to p do not increase anywhere, where $m = a_2 + a_0$. We claim that each of them is adjacent to p in the 5-sector Yao graph under the ordering

$$p, q_1, \ldots, q_m, \text{AnyOrder}(P_1).$$

As in the proof of Proposition 2, assume the contrary, namely that for some $j < i$, q_i is adjacent to q_j instead of p. First, suppose that q_i and q_j belong to different dual 5-sectors of p. This yields that $\measuredangle q_ipq_j > \frac{2\pi}{5} > \frac{\pi}{3}$. As earlier, we conclude that $\measuredangle q_ipq_j$ is not the smallest angle of the triangle q_ipq_j and q_iq_j is not its shortest side. Hence, $|q_ip| < |q_iq_j|$ and thus q_i should be adjacent to p instead of q_j, a contradiction.

Second, suppose that q_i and q_j belong to the same dual 5-sectors of p, say, $q_i, q_j \in P_2$. Then $q_j \in -s_2(q_i) \cap s_2(p)$. This intersection is a parallelogram with angles $\frac{2\pi}{5} < \frac{\pi}{2}$ at the vertices q_i and p. Therefore, q_ip is the diameter of this parallelogram, and thus $|q_ip| > |pq_j|$, a contradiction again. □

B.3 Upper bound on $d_3(n)$

Since $d_3(n)$ is clearly nondecreasing as a function of n, assume without loss of generality that $n = 6m$ for some $m \in \mathbb{N}$ and construct an n-element point set as follows. Pick a very small angle, say, $\alpha = \frac{\pi}{10m}$.

For $1 \le i \le m$, let a_i, b_i, c_i, d_i, e_i, and f_i be points on the unit circle whose angles with the x-axis equal to $i\alpha$, $-i\alpha$, $\frac{2\pi}{3} + i\alpha$, $\frac{2\pi}{3} - i\alpha$, $\frac{4\pi}{3} + i\alpha$, and $\frac{4\pi}{3} - i\alpha$, respectively, see Figure 4 (recall that $k = 3$). A small perturbation brings it in general position.

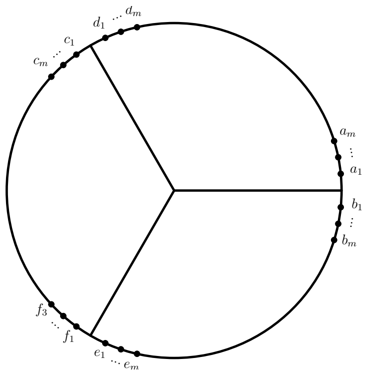

Figure 4: Both c_m and a_i belong to the first sector of f_1, both f_1 and a_i belong to the third sector of c_m, $c_m f_1$ is the shortest side of the triangle $a_i c_m f_1$.

To show that $d_3(n) \le 5 \left\lceil \frac{n}{6} \right\rceil - 1$, it suffices to show that for every ordering, the indegree of each vertex in the 3-sector Yao graph does not exceed $(n-1) - m$. Due to the symmetry of this configuration, it suffices to show this only for a_i, $1 \le i \le m$. Pick $1 \le j \le m$ and consider a triangle $a_i c_{m+1-j} f_j$. Note that both c_{m+1-j} and a_i belong to the first sector of f_j, and that both f_j and a_i belong to the third sector of c_{m+1-j}, see Figure 4. Moreover, it is not hard to see that $c_{m+1-j} f_j$ is the shortest side of the triangle $a_i c_{m+1-j} f_j$. Therefore, among c_{m+1-j} and f_j, the vertex that appears later cannot be adjacent to a_i. Since this holds for all $1 \le j \le m$, we conclude that the indegree of a_i does not exceed $(n-1) - m$ under each ordering, as desired.

C Clique numbers

C.1 Maximizing the largest clique: proof of Theorem 8

To prove the upper bound, consider a set P of n points on a generic line (not parallel to any of the ℓ_i). It is easy to see that each point contains all the others in (at most) two sectors, and thus its outdegree is at most two regardless of the ordering. Hence, the clique number of the k-sector ordered Yao graph is at most 3.

For $k = n = 3$, take a triangle centered at the origin and whose vertices are on ℓ_0, ℓ_1 and ℓ_2, see Figure 5. It it easy to see that for every order, the corresponding 3-sector ordered Yao graph contains only two edges, and thus it is triangle-free.

Now we prove the lower bounds. Our goal is to find three points p_1, p_2, p_3 in an arbitrary point set P such that one of them, say p_3, contains the other two

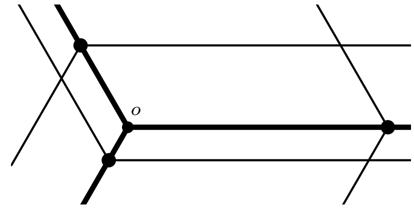

Figure 5: A set with a triangle-free 3-sector ordered Yao graph.

in different k-sectors. For such a triple, the k-sector ordered Yao graph corresponding to the order

$$p_1, p_2, p_3, \mathrm{AnyOrder}(P \setminus \{p_1, p_2, p_3\})$$

contains a triangle $p_1 p_2 p_3$.

First, consider the case when k is even. Among 3 arbitrary points, there is one, which is neither topmost, nor bottommost and thus does not contain the other two in the same k-sector, and we are done.

Similarly, if $k \ge 6$, then an arbitrary triangle from P has one angle at least $\frac{\pi}{3} \ge \frac{2\pi}{k}$, and thus its vertex contains the other two in different k-sectors.

If $k = 5$, we take three arbitrary points from P and translate the angles of their triangle Δ to a common origin, which divides the plane into 3 'cones' and 3 'complementary cones', see Figure 6. By the pigeonhole principle, either one of the cones contains at least one of the ℓ_i's, or at least one of the dual cones contains two. In the first case, we are done, because one of the three points has the other two in different 5-sectors. In the second case, we are also done, since one of the angles of Δ is larger than $\frac{2\pi}{5}$, and thus it must contain an ℓ_i.

Finally, suppose that $k = 3$, $n \ge 4$, and each point contains the others in one 3-sector. By the pigeonhole principle, some two of the points contain all the others in the same 3-sector, say, in s_0. However, for all $p_1, p_2 \in P$, if $p_2 \in s_0(p_1)$, then p_2 is higher than p_1, and thus $p_1 \notin s_0(p_2)$. This contradiction completes the proof.

C.2 Minimizing the largest clique: proof of Theorem 10

The upper bound is trivial: since the last $\left\lfloor \frac{k}{2} \right\rfloor$ of the k-sectors corresponding to each vertex belong to the lower half-plane, they do not contain the preceding points in the top-to-bottom ordering. Hence, all vertices have outdegree at most $\left\lceil \frac{k}{2} \right\rceil$ in the k-sector ordered Yao graph, and so the size of any clique is at most $\left\lceil \frac{k}{2} \right\rceil + 1$.

For the lower bound, we construct a set P of $\left\lceil \frac{k}{2} \right\rceil$ points in the plane such that every point contains all the others in pairwise distinct k-sectors. It is clear that no matter how we order these points, the cor-

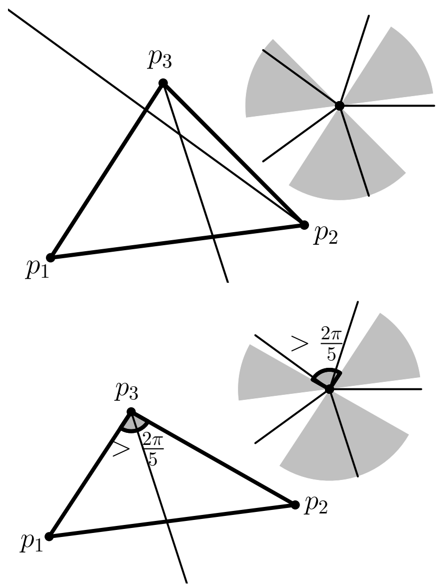

Figure 6: Illustration for the $k = 5$ case of Theorem 8. On the top, we can see an occurrence of the first subcase when there are cones (colored in grey) which contain at least one of the ℓ_i. On the bottom, we can see an occurrence of the second subcase when there is a complementary cone containing two of the ℓ_i, which also results in the cone on the opposing side containing an ℓ_i, or in other words, one of the p_i seeing the other two of the p_i in different sectors.

responding k-sector ordered Yao graph would be a clique.

To formally describe this construction, we utilize the standard bijection between the plane and the set of complex numbers $\mathbb{C}$. Let z be the $2k$-th root of 1 with the smallest positive argument, that is $z = \cos\frac{\pi}{k} + i \cdot \sin\frac{\pi}{k}$. Note that the directions of the rays $\ell_0, \ell_1, ..., \ell_{k-1}$ are $z^0, z^2, ..., z^{2k-2}$, respectively. Besides, note that the vector $z^{j_2} - z^{j_1} = z^{j_2} + z^{j_1+k}$ has direction $z^{(j_1+j_2+k)/2}$ for all $0 \le j_1 < j_2 \le 2k$.

In case of even k, put $p_j = z^{4j}$ for $0 \le j < \frac{k}{2}$. It is easy to verify that for each j_1, the directions of the vectors $p_{j_2} - p_{j_1}$, $0 \le j_2 < \frac{k}{2}, j_2 \ne j_1$, coincide with cyclically consecutive sector boundaries. Rotating this configuration by a sufficiently small angle about the origin moves these directions into the interiors of distinct cyclically consecutive k-sectors. That is $P = \{p_j \cdot z^{\varphi} : 0 \le j < \frac{k}{2}\}$ is the desired configuration for a sufficiently small φ.

In case $k = 2m+1$ and m is odd, we put $p_j = z^{4j}$ for $0 \le j < m$ and $p_m = z^{2k-3}$. We claim that for each fixed $0 \le j_1 < m$, the points p_{j_2}, $0 \le j_2 \le m, j_2 \ne j_1$ belong to the interiors of pairwise distinct k-sectors of p_{j_1}. Indeed, the direction of the vector $p_m - p_{j_1}$ is $z^{2j_1+k+(k-3)/2}$, and the latter exponent is an odd integer. Therefore, p_m lies on the bisector of some k-sector of p_{j_1}. Observe that $\frac{p_0}{p_m} = \frac{p_m}{p_{m-1}} = z^3$. Hence, we have $\measuredangle p_0 p_{j_1} p_m = \measuredangle p_m p_{j_1} p_{m-1} = \frac{3\pi}{2k}$, which equals $\frac{3}{4}$ of the k-sector angle. Thus p_{m-1}, p_m, and p_0 belong to 3 distinct cyclically consecutive k-sectors of p_{j_1}. (If $j_1 = 0$ or $j_1 = m-1$, then we only claim that the remaining two points belong to distinct k-sectors of p_{j_1}.) Similarly, $\frac{p_{j_2+1}}{p_{j_2}} = z^4$ implies that $\measuredangle p_{j_2} p_{j_1} p_{j_2+1} = \frac{2\pi}{k}$ for all $0 \le j_2 < m-1, j_2 \ne j_1, j_2+1 \ne j_1$, which equals the full k-sector angle. Therefore, all the points p_{j_2}, $0 \le j_2 \le m, j_2 \ne j_1$ belong to the interiors of pairwise distinct k-sectors of p_{j_1}, as claimed. Note that a rotation about the origin by a sufficiently small angle does not affect this property.

The last point p_m stands out in the sequence, and requires a separate analysis. Observe that the direction of the vector $p_j - p_m$ is $z^{2j+(k-3)/2}$, and the latter exponent is an even integer for all $0 \le j < m$. Hence, all the points p_j, $0 \le j < m$, lie on the boundaries of cyclically consecutive k-sectors of p_m. Rotating this configuration by a sufficiently small angle about the origin moves these points into the interiors of distinct cyclically consecutive k-sectors of p_m. That is, $P = \{p_j \cdot z^{\varphi} : 0 \le j \le m\}$ is the desired configuration for a sufficiently small φ.

The case $k = 2m+1$ and m is even is very similar to the previous one with the only difference being the multiplication by z. In other words, we put $p_j = z^{4j+1}$ for $0 \le j < m$ and $p_m = z^{2k-2}$. As before, we claim that for each fixed $0 \le j_1 < m$, all the points p_{j_2}, $0 \le j_2 \le m, j_2 \ne j_1$ belong to the interiors of pairwise distinct k-sectors of p_{j_1}. Indeed, the direction of the vector $p_m - p_{j_1}$ is $z^{2j_1+k+(k-1)/2}$, and the latter exponent is an odd integer. Therefore, p_m lies on the bisector of some k-sector of p_{j_1}, and the rest of the claim's proof goes as before. Finally, the direction of the vector $p_j - p_m$ is $z^{2j+(k-1)/2}$, and the latter exponent is an even integer for all $0 \le j < m$. Hence, all the points p_j, $0 \le j < m$, lie on the boundaries of cyclically consecutive k-sectors of p_m. As mentioned above, the set $P = \{p_j \cdot z^{\varphi} : 0 \le j \le m\}$ obtained by a sufficiently small rotation about the origin satisfies the desired condition.

D Edge numbers

For any point set and any order, the number of edges is the sum of the outdegrees of the points. And as we know, the outdegree of a point is simply the number of non-empty k-sectors at the moment of its addition. Thus, this is the only quantity we focus on in this section as the distances between points are irrelevant in this aspect. This also means that any statement for the edge number of ordered Yao graphs could also be stated for example for ordered theta graphs.

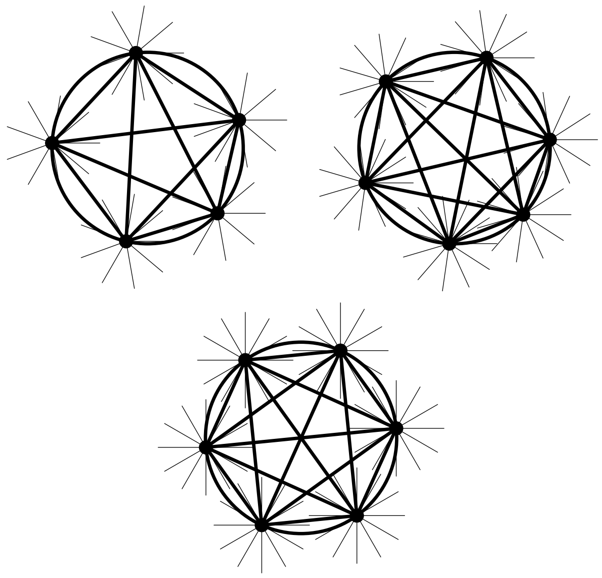

Figure 7: Examples for $k = 9, 11, 12$

D.1 Maximizing the number of edges: proof of Theorem 4

The lower bound is a simple corollary of Theorem 8. Indeed, if $k \neq 3$ and $n \geq 3$, then by Theorem 8, in every n-point set, there exists a point p_n that does not contain all the others in a single k-sector. Therefore, if this point appears the last, then its outdegree in the corresponding k-sector Yao graph would be at least 2. Delete this point from our set and recursively repeat the same argument while there are at least 3 points left. No matter how we order the remaining 2 points, the second one is adjacent to the first. As a result, we construct an ordering such that the corresponding k-sector ordered Yao graph contains at least $0 + 1 + 2 \cdot (n - 2) = 2n - 3$ edges, as desired. The case $k = 3$ is similar, with the only difference that we need at least 4 points left to recursively apply Theorem 8. As a result, we construct an ordering such that the corresponding 3-sector ordered Yao graph contains at least $0 + 1 + 1 + 2 \cdot (n - 3) = 2n - 4$ edges, as desired.

As for the upper bound, observe that for n points on a line (not parallel to the ℓ_i, so the construction is in general position), the two endpoints both have one sector which contains all the other points, while the rest has two, thus no ordering can generate more than $2n - 3$ edges, regardless of the value of $k \geq 2$. In case $k = 3$, Figure 10 provides a better construction: whatever ordering we take, A, B and C have outdegree at most 1, while all other points have outdegree at most 2. Combined with the fact that the first point always has outdegree 0, this gives an upper bound of $2n - 4$ on the number of edges.

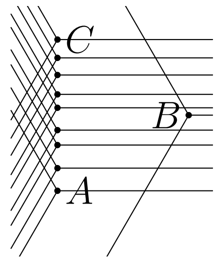

Figure 8: The points of the construction along with their incident rays.

D.2 Minimizing the number of edges: proof of Theorem 6

As noted earlier in the proof of Theorem 10, the upper bound $e_k(n) \leq n \cdot \lceil \frac{k}{2} \rceil$ is trivial. Indeed, the last $\lfloor \frac{k}{2} \rfloor$ of the k-sectors corresponding to each vertex belong to the lower half-plane, and so they do not contain the preceding points in the top-to-bottom ordering. Hence, all vertices have outdegree at most $\lceil \frac{k}{2} \rceil$ in the corresponding k-sector ordered Yao graph, and so the graph has at most $n \cdot \lceil \frac{k}{2} \rceil$ edges in total. For $k = 2$ this upper bound is trivially sharp for all P. Thus, in the following subsections, we assume that $k \geq 3$.

D.2.1 Upper bound: $e_k(n) \leq n \cdot \lceil \frac{k}{2} \rceil - \lceil \sqrt{n} \rceil \cdot \lfloor \frac{k+1}{4} \rfloor$ for $k \geq 3$

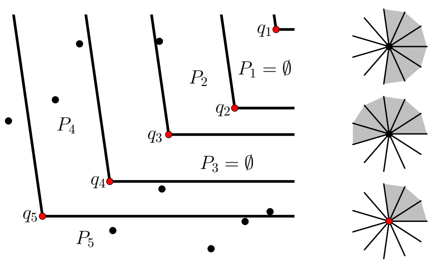

Figure 9: In the left subfigure, the points of Q are denoted by red and the $\ell_0(q_i)$ and the $\ell_{\lceil \frac{k}{4} \rceil}$ are also drawn. The point sets P_i are in between q_i and q_{i+1}. In the right subfigures, we can see the three cases of what a newly added point "sees": the sectors in grey are possibly non-empty, while the sectors in white are empty. The third one represents a q_i.

For any point p, let $x(p)$ and $y(p)$ be the Cartesian coordinates of p. Moreover, we consider the intersection of the x-axis with the line passing through p along the direction of $\ell_{\lceil \frac{k}{4} \rceil}$, and denote the x-coordinate of this intersection as $x'(p)$. Note that the general position hypothesis guarantees that $y(p) \neq y(q)$ and $x'(p) \neq x'(q)$ for $p \neq q$, although it is possible that $x(p) = x(q)$ unless $4|k$.

By the Erdős–Szekeres theorem, we can find a subset $Q = \{q_1, ..., q_m\}$ for some $m \geq \lceil \sqrt{n} \rceil$ with

$x'(q_i) \le x'(q_{i+1})$ and satisfying either $y(q_i) < y(q_{i+1})$ or $y(q_i) > y(q_{i+1})$ for all i. Without loss of generality, we assume the former case $y(q_i) < y(q_{i+1})$ is true and Q is maximal among all such subsets. We denote $P_i^+ = \{p \in P : x'(p) > x(q_i),\ y(p) > y(q_i)\}$ and $P_{m+1}^+ = P$ as a convention. Thus, we define $P_i = P_{i+1}^+ \setminus P_i^+$ and notice that $P_1, \dots, P_m$ form a partition of $P \setminus Q$ otherwise the maximality of Q would be violated.

Now, we determine an ordering on the points of P_i for a fixed i: First, take the points which have a larger y value than $y(q_i)$, ordered by their x'-values in a decreasing order; Then take the remaining points, ordered by their y values in decreasing order. Let us denote this ordering of P_i as $\mathrm{Order}(P_i)$. Thus, we define the ordering of P as follows:

$$q_1, \mathrm{Order}(P_1), q_2, \mathrm{Order}(P_2), \dots, q_m, \mathrm{Order}(P_m).$$

Finally, we bound the number of edges in this ordering. Indeed, the points preceding q_i have both a larger x'-value and a larger y-value coordinate than q_i, which means q_i has outdegree at most $\lceil \frac{k}{4} \rceil$. For each point in P_i, it only sees preceding points from either right or above (see Figure 9 right), so it has outdegree at most $\lceil \frac{k}{2} \rceil$. Therefore, we can conclude our upper bound using $m \ge \lceil \sqrt{n} \rceil$ and $\lfloor \frac{k+1}{4} \rfloor = \lceil \frac{k}{2} \rceil - \lceil \frac{k}{4} \rceil$.

D.2.2 Lower bound

We construct the n-element point set P as follows: Take a set of size n that contains the $\lfloor \sqrt{n} \rfloor \times \lfloor \sqrt{n} \rfloor$ grid and is contained in a $\lceil \sqrt{n} \rceil \times \lceil \sqrt{n} \rceil$ grid. Then apply a scaling transformation to make the gap of the grid slightly larger than 1, but the diameter of the set remains at most $\lceil \sqrt{2} \cdot \lceil \sqrt{n} \rceil \rceil$. Now make an even smaller perturbation of the point set ensuring that no isosceles triangles remain, neither two pairs of points whose segment is parallel to one of the ℓ_i, but the minimum distance among the points remains over 1 and the diameter of the point set remains under $\lceil \sqrt{2} \cdot \lceil \sqrt{n} \rceil \rceil$. The resulting set is our P.

We show that the number of edges is always at least $n \cdot \lceil \frac{k}{2} \rceil - O\left(\sqrt{n} \cdot k^2\right)$ in the ordered Yao graph associated with P regardless of the ordering. In particular, we fix an arbitrary ordering of P. Define multiset S containing vertices with outdegree less than $\lceil \frac{k}{2} \rceil$ and let the multiplicity of each vertex be the difference of $\lceil \frac{k}{2} \rceil$ and its outdegree, It suffices to prove that $|S| \le O\left(\sqrt{n} \cdot k^2\right)$ regardless of the ordering. Now let S_i be the set of points p for which both $s_i(p)$ and $s_{i+\lfloor k/2 \rfloor}(p)$ are empty when p is added. It is straightforward that $S \subseteq \bigcup_{i=0}^{k-1} S_i$, meaning $|S| \le \sum_{i=0}^{k-1} |S_i|$. Thus we only have to prove that $|S_i| \le O\left(\sqrt{n} \cdot k\right)$ for all $i = 0, ..., k-1$.

We make the following simple observation.

Lemma 12 *Let p be a point in the plane and r_1, r_2, r_3 and r_4 be distinct rays starting in p counterclockwise in the aforementioned order with $\angle r_i r_{i+1}$ being called α_i and the closed cone defined by r_i and r_{i+1} by C_i (the indices being counted $\bmod 4$). If $\alpha_1 = \alpha_3$ and both α_2 and α_4 are smaller than π, then for any strip σ of width 1 perpendicular to to the angle bisector of C_2 (and C_4) and containing p, all points of $\sigma \cap (C_2 \cup C_4)$ have distance at most $\max\left\{\frac{1}{\cos(\alpha_2/2)}, \frac{1}{\cos(\alpha_4/2)}\right\}$ from p.*

Next, we define strips $\sigma_{i,j}$ for $i = 1, \dots, k-1$ and $j = 1, \dots, t_i$ for some $t_i < \lceil \sqrt{2} \cdot \lceil \sqrt{n} \rceil \rceil$. Consider the direction that is perpendicular to the line bisecting ℓ_i and $\ell_{i+\lfloor k/2 \rfloor+1}$, and denote the minimal strip containing P parallel to this direction as σ_i. Notice that the diameter of P is at most $\lceil \sqrt{2} \cdot \lceil \sqrt{n} \rceil \rceil$, we can cover σ_i by at most this number of strips of width 1 along the same direction, and denote then as $\sigma_{i,j}$. See Figure **??**.

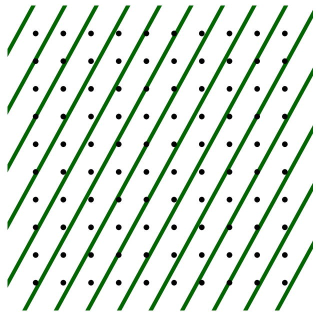

Figure 10: The $\sigma_{i,j}$'s for some fixed i.

Now, it is enough to prove that $S_i \cap \sigma_{i,j} = O(k)$ for any i, j and then sum up these numbers for a fixed i. Let p be the leftmost point and q be the rightmost point among $S_i \cap \sigma_{i,j}$. (When $\sigma_{i,j}$ is vertical, we instead take p as the lowest and q as the highest.) We apply Lemma 12 with $r_1 = \ell_i$, $r_2 = \ell_{i+1}$, $r_3 = \ell_{i+\lfloor k/2 \rfloor}$, $r_4 = \ell_{i+\lfloor k/2 \rfloor+1}$, and $\sigma = \sigma_{i,j}$. As a consequence, we conclude that the maximum distance between p and any point in $\sigma_{i,j} \setminus \left(s_i(p) \cup s_{i+\lfloor k/2 \rfloor}(p)\right)$ is at most

$$D := \frac{1}{\cos\left(\pi \cdot (\lceil k/2 \rceil - 1)/k\right)}.$$

A similar statement holds for q by the same argument. Observe that either $q \in \sigma_{i,j} \setminus \left(s_i(p) \cup s_{i+\lfloor k/2 \rfloor}(p)\right)$ or $p \in \sigma_{i,j} \setminus \left(s_i(q) \cup s_{i+\lfloor k/2 \rfloor}(q)\right)$, otherwise the one that comes later in the ordering could not be in S_i. Hence, the distance between p and q is at most D. According to how p and q are chosen, this means that $S_i \cap \sigma_{i,j}$ is contained in a rectangle of size $1 \times D$. Also, since we scaled the gap of the grid to be larger than 1, such a rectangle can contain at most $2\lceil D \rceil$ grid points (hence also points from P).

Finally, it is easy to check that $D \le Ck$ for some absolute constant C using trigonometry and calculus, and thus conclude the proof.