On geodesic disks enclosing many points

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Given a set S of n points in the plane in general position—no three of them are collinear and no four of them are cocircular—there always exists two points $u, v \in S$ such that any disk that contains u and v also contains a constant fraction of S. Neumann-Lara and Urrutia [6] proved that this constant fraction is at least $\lfloor \frac{n-2}{60} \rfloor$. This bound was improved in a series of papers [4, 5] culminating in the best known lower bound of $\frac{\sqrt{3}-1}{2\sqrt{3}}n + O(1) \approx \frac{n}{4.7}$, see [2, 3, 8]. Edelsbrunner et al. [3] were the first to show this using techniques related to the k-th order Voronoi diagram. Ramos and Viaña [8] used known results about j-facets of point sets in $\mathbb{R}^3$ to also prove that there is always a pair of points such that any disk through them has at least $\frac{n}{4.7}$ points both inside and outside the disk. Hayward et al. [5] gave an upper bound by constructing a set of n points such that for every pair of points u, v, there exists a disk through u, v containing less than $\lceil \frac{n}{4} \rceil$ points, thereby showing that $\lceil \frac{n}{4} \rceil + 1$ is an upper bound for the problem. In addition, they studied the problem for point sets in convex position, giving a tight bound of $\lceil \frac{n}{3} \rceil + 1$. Another version of the problem is to consider disks in the plane having the points u and v as diametral endpoints [1]. In this case, a tight bound of $\lceil \frac{n}{3} \rceil + 1$ for both the convex and non-convex cases was shown. A colored version of the problem has been studied in the literature [2, 7]. Given a set S of $\frac{n}{2}$ red and $\frac{n}{2}$ blue points in the plane, there always exists a bichromatic pair of points $u, v \in S$ such that any disk containing u and v contains a constant fraction of points. This fraction was proved to be at least $\frac{\sqrt{2}-1}{2\sqrt{2}}n - o(n) \approx \frac{n}{6.8}$, see [2].

The problem of studying the largest number of points contained in any geodesic disk through two points inside a polygon P is distinct from the planar case. A geodesic disk of radius r in P can be strictly contained in a Euclidean disk of radius r. As such, in this setting, it is unclear whether there always exists a pair of points such that any geodesic disk through them can contain the same number of points as in the Euclidean setting. In this paper, we present an overview of key properties of shortest paths in P, emphasizing both their similarities to and differences from the Euclidean case. It is these differences that pose challenges we address to extend the results to the geodesic setting. Using these properties, we extend the lower bounds from the Euclidean case [2, 3, 8] to the geodesic setting, see Table 1. We also show that the techniques in [1] and [5], where diametral disks and points in convex position are, respectively, considered, can be generalized to the geodesic setting.

Setting	General case	In-out	Bichromatic
Euclidean	$\lfloor \frac{n}{4.7} \rfloor$ [3]	$\lfloor \frac{n}{4.7} \rfloor$ [8]	$\lfloor \frac{n}{6.8} \rfloor$ [2]
Geodesic	$\lfloor \frac{n}{5} \rfloor$	$\lfloor \frac{n}{5.2} \rfloor$	$\lfloor \frac{n}{11.1} \rfloor$

Table 1: Summary of some of our results, comparing the Euclidean and geodesic settings.

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