A PTAS for the Unit Disk Uniform Multi-Cover Problem

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1 Problem

The disk cover problem is a fundamental optimization problem with wide applications, including wireless sensor networks. In this paper, we present the first polynomial-time approximation scheme (PTAS) for the unit disk uniform Multi-Cover problem (UDUMC) in the Euclidean plane. We assume all disks have the same radius, that without loss of generality we can assume to be 1. UDUMC is NP-hard even when $m = 1$ [2]. In the UDUMC, we are given a set of n points in the plane, called *targets*, $T = \{t_1, t_2, \ldots, t_n\}$, along with a positive integer parameter m. The goal is to place a minimum number of unit disks such that each target is covered by at least m disks.

2 Algorithm

First, our algorithm extends the PTAS for UDC [1] to UDUMC by applying its shifting strategy. Following the shifting strategy, we partition the input region I–the minimal rectangle enclosing all targets–into a set of squares $\{Q\}$, each of side length $2c$, where c is a constant and $m \in N^+$. This yields local subproblems $A(Q)$, defined as follows.

Definition 1 (Problem $A(Q)$) *: Given a square Q $T(Q)$ with k targets, compute the minimum number of unit disks needed to fully cover each target in Q at least m times.*

Second, to reduce the computational burden, each local problem $A(Q)$ is further broken into simpler subproblems, called $A(Q, c)$. An approximate solution is obtained by solving one simple problem using exhaustive enumeration, and the result is reused for all, significantly reducing total complexity.

Theorem 2 *There exists a PTAS for the UDUMC. The running time is $O(c^2 n^{c^7})$ and the approximation ratio is $O(1 + 1/c)^3$, where $c > 6$ is an integer number.*

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3 An approximation algorithm for $A(Q)$

Solving $A(Q)$ efficiently is the core of our approximation scheme. The high-level idea is to decompose $A(Q)$ into subproblems $A(Q, c)$ of the same type, where each target must be covered $c^2\lceil\sqrt{2}c\rceil^2$ times. A feasible solution is obtained by enumerating a representative case. However, direct enumeration lacks a structure for bounding the approximation error. To address this, we reorganize the enumeration into an iterative framework that progressively increases the coverage count for each target while strictly controlling both lower and upper bounds.

The algorithm begins by enumerating all feasible coverings in which each target is covered at least once and at most c times. In each subsequent iteration i $(2 \le i \le c^2\lceil\sqrt{2}c\rceil^2)$, a new set of candidate subsets is generated and merged with each coverage scheme from the previous iteration. A merged solution is retained if it ensures that every target is covered between i and $i + c - 1$ times. This process continues until the $c^2\lceil\sqrt{2}c\rceil^2$-th iteration, where the final coverage schemes guarantee that each target is covered at least $c^2\lceil\sqrt{2}c\rceil^2$ times and at most $c^2\lceil\sqrt{2}c\rceil^2 + c - 1$ times.

For time complexity, computing $A(Q, c)$ takes $O(k^{c^7})$ time, yielding an overall complexity of $A(Q)$ $O(k^{c^7})$. With a shifting strategy, UDUMC runs in $O(c^2 n^{c^7})$ time. For approximation ratio, each subproblem $A(Q, c)$ yields a solution with approximation ratio $O\left(1 + \frac{1}{c}\right)$, then, merging these into $A(Q)$ introduces another $O\left(1 + \frac{1}{c}\right)$ error. So the overall ratio is $O\left(1 + \frac{1}{c}\right)^2$. Applying the shifting strategy adds an independent factor of $O\left((1 + \frac{1}{c})^2\right)$, resulting in a final approximation ratio of $O\left((1 + \frac{1}{c})^4\right)$.

References

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