On the bisector of two low degree curve segments in the plane

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Abstract

We show how to compute an exact representation of the bisector for two curve segments in the plane whose parametrization is a rational function of degree smaller than or equal to 2. It is a piecewise curve composed of at least one component and with no more than seven components.

1 Introduction

Consider O_1 and O_2 two curve segments in $\mathbb{R}^2$, presented parametrically by $\mathbf{O}_1(u), u \in I_1$, and $\mathbf{O}_2(t), t \in I_2$, with I_1 and I_2 closed intervals in $\mathbb{R}$. Their bisector $\mathcal{B}(O_1, O_2)$ is defined as the equidistant set of points from the two objects. The computation of a representation for the bisector is not an obvious task from this definition. Indeed, even if the parametric object is given with a regular and proper parametrization, it should be noted that the distance function

$$d : (B, O) \mapsto \inf_{u \in I} \|B - \mathbf{O}(u)\|, \qquad (1)$$

is not always differentiable with respect to the point B, and a minimum of the distance function could be achieved at more than one parameter value. To overcome these difficulties, the notion of untrimmed bisector was introduced (see [7], [8]).

Definition 1 *The untrimmed bisector of O_1 and O_2 is defined as the set of centres of circles which are tangent to O_1 and O_2 simultaneously.*

This definition does not imply the same minimum distances measured from the two objects, in the presence of critical shapes on the objects (singular, inflection or self-intersection points). There are some extraneous parts that should be trimmed in order to obtain the searched bisector.

We show here how to compute the representation for the bisector of two curve segments whose parametrization is a rational function of degree smaller than or equal to 2 (i.e. the degree of the polynomials involved is smaller than or equal to 2). This can be applied to determining the medial axis or the Voronoi diagram for objects constructed from segments of curves (see [1, 3, 9, 10]).

The output of our method will include:

1. The exact representation for the bisector of an endpoint A_i and open curve segment c_j denoted by $\mathcal{B}(A_i, c_j)$ is computed following [7].
2. The bisector of endpoints A_i and A_j denoted by $\mathcal{B}(A_i, A_j)$ is a half line or a line segment whose representation is trivial.
3. The exact representation for the bisector of two open curve segments c_i and c_j denoted by $\mathcal{B}(c_i, c_j)$ is computed either from the formula (7) below in the next section, or from an implicit representation $F(x, y) = 0$ (see [2]) of the bisector of the two corresponding curves.

If the degree of the parametrization is higher then the degree of the bisector gets too big for practical computations in case exact computations are to be performed. In [5] it is shown that for two cubic curves, the bisector is an algebraic curve of degree 46.

2 Computing the untrimmed bisector of two plane rational curves

The algebraic representation of the untrimmed bisector of two regular plane rational curves s and r with parametrizations, respectively $s(u)$ and $r(t)$, is described as follows:

A point $\mathbf{B} = (X, Y)^T \in \mathbb{R}^2$ is in the untrimmed bisector of the curves s and r if it satisfies the following system of equations (see [4, 5, 6]):

- The point $\mathbf{B}$ is in the normal lines of s and r, at $\mathbf{s}(u)$ and $\mathbf{r}(t)$, respectively:

$$\begin{aligned} \langle (X, Y) - \mathbf{s}(u), \mathbf{s}'(u) \rangle &= 0, \\ \langle (X, Y) - \mathbf{r}(t), \mathbf{r}'(t) \rangle &= 0, \end{aligned} \qquad (2)$$

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where $\mathbf{s}'$ and $\mathbf{r}'$ denote the derivatives of s and r.

- The point $\mathbf{B}$ is at equal distance from $\mathbf{s}(u)$ and $\mathbf{r}(t)$:

$$\begin{aligned}\langle (X,Y), 2\,(\mathbf{r}(t)-\mathbf{s}(u))\rangle + \\ +\|\mathbf{s}(u)\|^2 - \|\mathbf{r}(t)\|^2 = 0.\end{aligned} \tag{3}$$

Equations (2) can be written in matrix form $\mathbf{A}\,\mathbf{B} = \mathbf{V}$, with

$$\mathbf{A} = \begin{bmatrix} \mathbf{s}'_x(u) & \mathbf{s}'_y(u) \\ \mathbf{r}'_x(t) & \mathbf{r}'_y(t) \end{bmatrix}, \mathbf{V} = \begin{bmatrix} \langle \mathbf{s}(u), \mathbf{s}'(u)\rangle \\ \langle \mathbf{r}(t), \mathbf{r}'(t)\rangle \end{bmatrix} \tag{4}$$

Our goal is to compute a parametrization of the bisector of the curves s and r in terms of one parameter, either u or t. Our approach consists in:

- First solve the system (4) for $\mathbf{B}$ in terms of u and t, using Cramer's rule: For $\det(\mathbf{A}) \neq 0$, we have:

$$\mathbf{B}(u,t) = \mathbf{A}^{-1}\mathbf{V}, \tag{5}$$

  and substituting $\mathbf{B}(u,t)$ in (3), we obtain the equation:

$$\begin{aligned}F(u,t) = \langle \mathbf{B}(u,t), 2(\mathbf{r}(t)-\mathbf{s}(u))\rangle + \\ +\|\mathbf{s}(u)\|^2 - \|\mathbf{r}(t)\|^2 = 0.\end{aligned} \tag{6}$$

- Then, express one of the parameters, say u in terms of t, from the equation (6). Since there might be more than one solution, we get $u_i = u_i(t),\ i = 1,\ldots,m$.

- Finally, substitute u by $u_i(t)$ in $\mathbf{B}(u,t)$, for each solution, and obtain the parametrization of the untrimmed bisector of the form:

$$\mathbf{b}_i(t) = \mathbf{B}(u_i(t),t) = [x_i(t), y_i(t)]^T, \tag{7}$$

  where $x_i(t), y_i(t)$ are in general non-rational.

For given values of the parameters u and t, a bisector point can be computed from the formula (5). Then for two corresponding footpoints $p_1 = s(u_0) \in s$ and $p_2 = r(t_0) \in r$, their corresponding bisector point is given by $\mathbf{B}(u_0,t_0)$.

Note that the bisector of a point and a rational curve has a rational parametrization: the process of computing the bisector and trimming its extraneous part is presented for example in [7]. There are several pairs of geometric objects possessing a rational bisector, but in general it is very difficult to have a criterion for the rationality of the bisector, and very few generic configurations of objects with rational bisector are known.

The bisector of two planar rational curves is not a rational curve in general (see [8]). The algebraic representation is of high degree and determining the trimming of the extraneous part is not evident. An approximate representation can also be used in the non rational case (see [6]). An algebraic approach is given in [2] to compute an algebraic (rational and non rational) parametrization for the bisector for some particular curves. The trimming process is also shown.

3 The bisector of two low degree curve segments

Consider two curve segments s_i and s_j whose parametrization is a rational function of degree smaller than or equal to 2. There are two possible configurations to take into account: the curve segments share one endpoint, or the curve segments are disjoint. The bisector will be constructed as a combination of the bisectors of a couple of distinct objects of s_i and s_j: point-point, point-curve and curve-curve. The process involves, for each endpoint of s_i, to determine, if possible, the corresponding footpoint on s_j. In this way we identify and properly store the various components forming the bisector. The determination of a footpoint will be done by computing the corresponding parametric values through the equation (6).

Remark 1 The equation (6) is an equivalent representation for the point $\mathbf{B}(u,t)$ of the bisector of the two curve segments in terms of parameters u and t (see [2]). More precisely, the solution of (6) represents the set of all couples of footpoints on $s_i(u)$ and $s_j(t)$ that share a bisector point $\mathbf{B}(u,t)$ (see [4, 6]).

Proposition 1 *Let $s_i(u)$, $u \in [0,1]$ and $s_j(t)$, $t \in [0,1]$ be two curve segments with their respective endpoints $A_i = s_i(0)$, $B_i = s_i(1)$, $A_j = s_j(0)$ and $B_j = s_j(1)$. The set of solutions of the equation (6) on the boundary of the parameter domain $[0,1]\times[0,1]$, if non empty, corresponds to the couples of parameter values of an endpoint of a segment and its corresponding footpoint (probably more than one) on other segment curve.*

Remark 2

1. An endpoint of a segment and its corresponding footpoint on the other segment correspond to a point of the bisector of s_i and s_j where there is a change of the component as indicated in Figure 1 .

2. For two curve segments, in general not all points of a curve segment have a corresponding footpoint on the other curve segment, and precisely if the equation (6) has no solution there is no curve-curve bisector component involved in the bisector of the two curve segments. It consists essentially of the point-point and point-curve bisector components as indicated in Figure 2 .

In what follows $\star$ will denote 0 or 1, and $\overline{\star} = 1$ if $\star = 0$ and vice-versa.

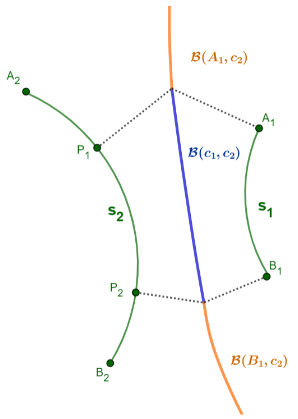

Figure 1: A_1 and B_1 with respective corresponding footpoints p_1 and p_2 on c_2: $\mathcal{B}(c_1, c_2)$ in blue, $\mathcal{B}(A_1, c_2)$ and $\mathcal{B}(B_1, c_2)$ in orange.

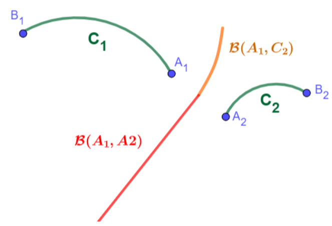

Figure 2: No corresponding footpoints. $\mathcal{B}(A_1, A_2)$ in red and $\mathcal{B}(A_1, c_2)$ in orange.

Definition 2

1. *An endpoint of a segment is said Free-Point (FP) if it has no corresponding footpoint on the other segment sharing curve-curve bisector points.*

2. *An endpoint of a segment is said Non-Free-Point-type1 (NFP1) if it corresponds to a solution couple $(u_0, \star)$ (or $(\star, t_0)$) where u_0 (or t_0), is a unique value corresponding to a unique footpoint on the other segment sharing curve-curve bisector points.*

3. *An endpoint of a segment is said Non-Free-Point-type2 (NFP2) if it corresponds to a solution couple $(u_0, \star)$ (or $(\star, t_0)$) where u_0 (or t_0) has two values, i.e. an endpoint with exactly two corresponding footpoints on the other segment sharing curve-curve bisector points.*

The main theorem of this paper establishes that the upper bound for the number of components of the bisector of two curve segments is seven. The proof of this theorem requires to introduce three lemmas.

Lemma 1 describes the bisector of a point P and a curve segment $s = \{A, c, B\}$, where A and B are the endpoints, and c is the open curve segment (see Figure 3).

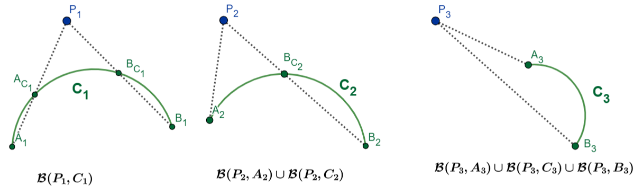

Figure 3: The interactions of a point and the elements of a curve segment when computing their bisector.

Lemma 2, for two disjoint curve segments $s_i = \{A_i, c_i, B_i\}$ and $s_j = \{A_j, c_j, B_j\}$, determines the domain where A_i interacts with s_j, in the cases where A_i is a FP or a $NFP1$.

Lemma 3 gives the number of bisector components generated by different types of endpoints (see Figures 4 and 5).

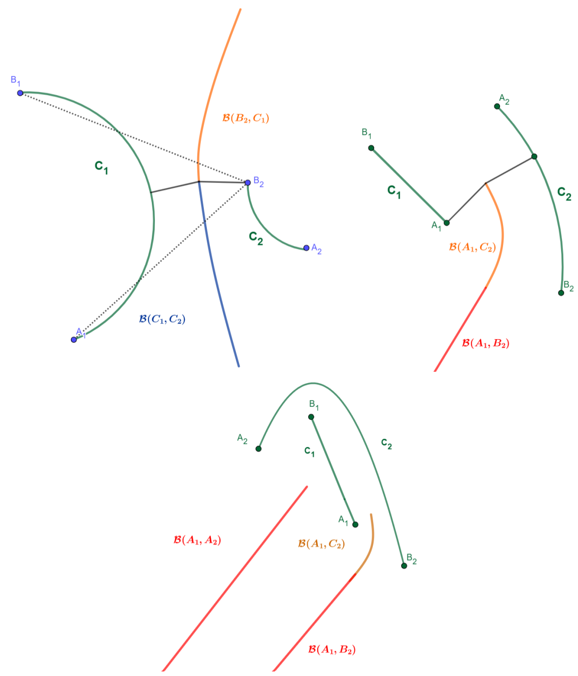

Figure 4: NPF1 configuration generating one, two and three components: point-curve bisector in orange and point-point bisector in red.

The following theorem specifies the maximum number of components appearing in the bisector of two curve segments whose parametrization is a rational function of degree smaller than or equal to 2.

Theorem 3 *Let s_i and s_j be two disjoint curve segments whose parametrization is a polynomial of degree smaller than or equal to 2. The bisector of s_i and s_j is composed of at least one component and with no more than seven components.*

Figure 6 shows two examples where the seven components case is achieved.

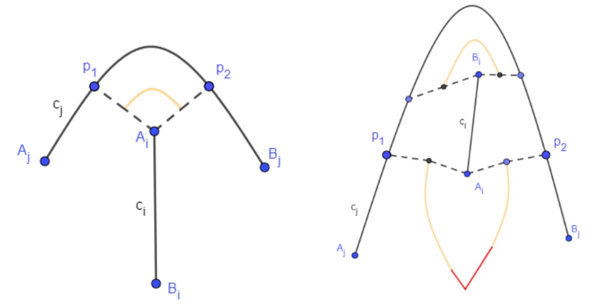

Figure 5: NPF2 configuration with a single and four bisector components: point-point bisector in red and point-curve in orange.

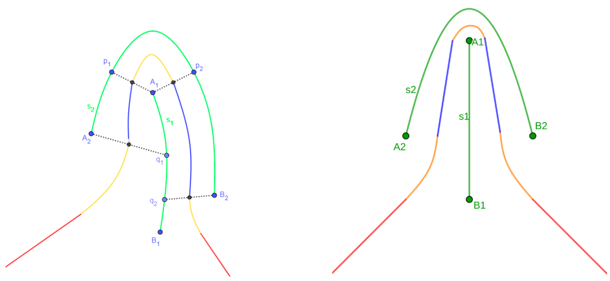

Figure 6: The two curve segments with one NFP2 for one curve segment and one NFP1 for the other: point-point bisectors in red and point-curve bisectors in orange

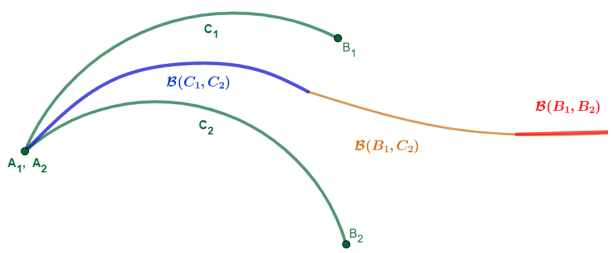

Figure 7: The bisector of two curve segments with the same endpoints.

3.1 The case of sharing one common endpoint

Assume $A_i = A_j$. In this case, the solution of the equation (6) is one or a couple of solutions of the parameter values: the first one corresponding to the (two) same endpoints, and the second one is corresponding to a free endpoint of one curve segment (s_i) and its corresponding footpoint on the other curve segment (s_j).

For the case of two curve segments sharing one endpoint, the bisector is composed by 1, 2 or 3 components (see, for example, Figure 7).

When s_i and s_j have exactly the same two endpoints ($A_i = A_j$ and $B_i = B_j$), the bisector has exactly one component, a curve-curve bisector.

4 Conclusions

A new approach is presented to compute the exact representation of the bisector of two low degree curve segments. The computed representation for each bisector component is either a rational parametrization, or a non rational parametrization (involving square roots), or a semi-algebraic representation (involving a real algebraic curve presented implicitly).

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