Containment results on points and spheres

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Abstract

Let S be a set of n points in general position in $\mathbb{R}^d$. We show several containment results for points from S in spheres determined by $d+1$ points of S. Among them, we prove a Delaunay-type criterion for point sets in $\mathbb{R}^3$. Also, we show bounds on the expected number of points from S contained in a sphere determined by four points chosen uniformly at random from $S \subset \mathbb{R}^3$. A tight upper bound construction is provided, obtained by inversion of points on the moment curve. We also show a lower bound and prove that it is best possible for $n \leq 7$. In order to do so, we solve the recurrence relation $T(n) = \left\lceil \frac{n}{n-5} T(n-1) \right\rceil$ with base case $T(7) = 29$. This is of independent interest, since most recurrence relations of this type seem not to have a solution in closed form.

1 Introduction

Let S be a set of $n \geq d+2$ points in general position in $\mathbb{R}^d$, $d \geq 2$, meaning no m of them lie on a $(m-2)$-dimensional flat for $m = 2, 3, ..., d+1$ and no $d+2$ of them lie on the same $(d-1)$-sphere. We show several containment results for points from S in the open balls having as boundary spheres determined by $d+1$ points from S. With a slight abuse of notation, usual in the literature, in the following we will say *sphere* instead of *open ball* for this containment relationship.

First, we prove a Delaunay-type criterion for point sets in $\mathbb{R}^3$. The well-known *empty circle property* of the Delaunay triangulation in $\mathbb{R}^2$, see e.g. Lemma 9.4 in [4], states that given a set $S = \{a, b, c, d\}$ of four points in convex position in the plane, then exactly two of the four circles passing through three points of S contain the fourth point of S; and if the line passing through two points a and b of S separates c and d, then the circle through a, b, c contains d if and only if the circle through a, b, d contains c. This criterion is commonly used to characterize the Delaunay triangulation of a set S of n points in $\mathbb{R}^2$ as the set of triangles with vertices from S, whose circumcircles are empty of other points from S. Delaunay in his paper [7] from 1934 stated this more generally for $\mathbb{R}^n$, $n \geq 2$. We obtain a statement similar to the Delaunay criterion, for five points in $\mathbb{R}^3$, given in Section 2.

Second, we study the expected number $\mathbb{E}(X_{S,d})$ of points from S that are contained in the sphere passing through $d+1$ different points from S, chosen uniformly at random. $\mathbb{E}(X_{S,d})$ is determined by the vector $(s_0, s_1, \ldots, s_{n-d-1})$, where s_k is the number of spheres passing through $d+1$ points of S that enclose exactly k other points from S, for $k = 0, \ldots, n-d-1$. Clearly, $\sum_{k=0}^{n-d-1} s_k = \binom{n}{d+1}$. Then,

$$\mathbb{E}(X_{S,d}) = \frac{\sum_{k=0}^{n-d-1} k \cdot s_k}{\binom{n}{d+1}}. \tag{1}$$

The expression $\sum_{k=0}^{n-d-1} k \cdot s_k$ can also be interpreted as the number of (p, Q) pairs, such that p is a point of S, and Q is a sphere induced by $d+1$ points of $S \backslash \{p\}$ containing p in its interior. In dimension $d = 2$, $\mathbb{E}(X_{S,2})$ is equivalent to the *rectilinear crossing number* [29] of S, denoted $\overline{cr}(S)$, via the known relation, first obtained by Urrutia [31], see also [12]:

$$\sum_{k=0}^{n-3} k \cdot s_k = \binom{n}{4} + \overline{cr}(S). \tag{2}$$

In this paper we mainly focus on dimension $d = 3$. We define $S_n = \min \sum_{k=0}^{n-4} k \cdot s_k$, where the minimum is taken over all sets S of n points in general position in $\mathbb{R}^3$.

In Section 3 we show a lower bound of $S_n \geq 2 \left\lfloor \frac{\binom{n}{5}}{5} \right\rfloor + \binom{n}{5} - 2 \left\lfloor \frac{n}{25} \right\rfloor$ for each $n \geq 5$. We have found point sets showing that this bound is best possible for $n \leq 7$; i.e., $S_5 = 1$, $S_6 = 8$, and $S_7 = 29$. Other found point sets show that $S_8 \leq 80, S_9 \leq 189,$ and $S_{10} \leq 376$.

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To prove the bound on S_n, we present a solution in closed form of the recurrence relation

$$T(n) = \left\lceil \frac{n}{n-a} T(n-1) \right\rceil \quad \text{for } n > b, \text{ and } T(b) = c,$$

with $a = 5$, $b = 7$, $c = 29$. Interestingly enough, Conway et al. [6] needed to solve, for a different problem, the case with $a = 3$, $b = 4$, $c = 1$, and stated that most recurrence relations of this shape, for given integers a, b, c, seem not to have a solution in closed form, leaving as an open problem to characterize those which do.

We will consider all five-tuples of points from S. For a set $\mathcal{F}$ of five points in $\mathbb{R}^3$ we say that $\mathcal{F}$ is of Type A if $s_0 = 4$ (then $s_1 = 1$), of Type B if $s_0 = 3$ (then $s_1 = 2$), and of Type C if $s_0 = 2$ (then $s_1 = 3$). Calling A, B, C the number of five-tuples of each type we can write

$$\sum_{k=0}^{n-4} k \cdot s_k = 1 \cdot A + 2 \cdot B + 3 \cdot C. \tag{3}$$

Note that these are the only possible types, because each sphere counted by s_0 corresponds to a simplex of the Delaunay triangulation of $\mathcal{F}$ and the number of simplices in a triangulation of n points with h of them on the boundary of the convex hull is between $n-3$ and $\binom{n-1}{2} - h + 2$ [11]. In particular, a set of five points in non-convex position is of Type A, whereas there are two types, B and C, of sets of five points in convex position.

In Section 4 we show that for a set S of n points on the moment curve in $\mathbb{R}^3$, all its five-tuples of points are of Type B. Then, among all sets S of n points in convex position, points on the moment curve minimize $\sum_{k=0}^{n-4} k \cdot s_k$. Let us also remark that the order type [13] of a point set does not determine the types A, B, C of all of its five-tuples of points. In particular, there are cyclic polytopes, i.e., point sets that have the same order type as a set of points on the moment curve, not all whose five-tuples are of Type B. We also prove that there exist sets of n points all of whose five-tuples are of Type C and thus, by (3), maximize $\sum_{k=0}^{n-4} k \cdot s_k$ among all sets of n points in general position in $\mathbb{R}^3$. Interestingly, these point sets are obtained by applying inversion to the points on an arc of the moment curve.

Finally, in Section 5 we consider sets S of points in $\mathbb{R}^d$, $d \geq 2$. Let $P_d(S)$ be the probability that the sphere passing through $d+1$ points chosen uniformly at random from S, contains another point chosen uniformly at random from the remaining points of S. We define $P_d(n)$ as the minimum of $P_d(S)$ among all sets S of n points in general position in $\mathbb{R}^d$, and $P_d^* = \lim_{n\to\infty} P_d(n)$. We prove that this limit exists for each fixed dimension d. For $d = 2$, we observe that P_2^* is equivalent to the *rectilinear crossing number constant* $\overline{\nu}^*$, see e.g. [30], namely $P_2^* = \frac{1+\overline{\nu}^*}{4}$ by using Equation (2). Then it is not surprising that the proof for existence of $\overline{\nu}^*$ from [30] extends smoothly to a proof for existence of P_d^* for $d > 2$. For dimension $d = 3$, we show the lower bound $P_3^* \geq \frac{7}{25}$. Other research on containment results on points and spheres was mainly carried out in dimension $d = 2$, see e.g. [2, 5, 10, 15, 16, 21, 25], with others, but noticeably fewer, for $d \geq 3$, see e.g. [3, 8, 9, 24, 27]. Due to lack of space, proofs are omitted.

2 A Delaunay-type criterion in $\mathbb{R}^3$

We say that a plane π separates two points d and e, if d and e do not lie in the same (closed) half-space bounded by π. We say that a triangle $\Delta(a, b, c)$ with vertices a, b, and c separates two points d and e, if the plane π passing through a, b, and c, separates d and e. We denote the sphere passing through four points a, b, c, and d with $\bigcirc(a, b, c, d)$.

Lemma 1 *Let $S = \{a, b, c, d, e\}$ be a set of five points in general and convex position in $\mathbb{R}^3$, such that the plane π passing through a, b, c separates d and e. Then the sphere $\bigcirc(a, b, c, d)$ contains e in its interior if, and only if, the sphere $\bigcirc(a, b, c, e)$ contains d in its interior.*

Theorem 2 *Let $S = \{a, b, c, d, e\}$ be a set of five points in general and convex position in $\mathbb{R}^3$, such that triangle $\Delta(a, b, c)$ separates d and e, and triangle $\Delta(a, d, e)$ separates b and c. Then, exactly two of the four spheres $\bigcirc(a, b, c, d)$, $\bigcirc(a, b, c, e)$, $\bigcirc(a, d, e, b)$ and $\bigcirc(a, d, e, c)$ contain the remaining point of S in its interior. Furthermore, $\bigcirc(a, b, c, d)$ contains e if, and only if, $\bigcirc(a, b, c, e)$ contains d; and $\bigcirc(a, d, e, b)$ contains c if, and only if, $\bigcirc(a, d, e, c)$ contains b.*

Remark 1 *Let $S = \{a, b, c, d, e\}$ be a set of five points in general and convex position in $\mathbb{R}^3$. It follows from Radon's lemma that we can relabel the points from S such that triangle $\Delta(a, b, c)$ separates d and e, and triangle $\Delta(a, d, e)$ separates b and c.*

3 A lower bound on the expected number of points in a sphere

Theorem 3 *For $n \geq 5$,*

$$S_n \geq 2\left\lfloor \frac{\binom{n}{5}}{5} \right\rfloor + \binom{n}{5} - 2\left\lfloor \frac{n}{25} \right\rfloor,$$

with equality for $5 \leq n \leq 7$. In particular, $S_5 = 1$, $S_6 = 8$, and $S_7 = 29$.

The equality for $n = 5$ is attained for a Type A set of five points. The equalities for $n = 6, 7$ are obtained by the example point sets and by case analysis on the number of points on the convex hull of a generic point set, together with some geometric arguments. For $n > 7$, the inequality follows from Lemmas 4, 5 and 6, and by using induction on n.

Lemma 4

$$S_n \geq \left\lceil \frac{n}{n-5} S_{n-1} \right\rceil .$$

Lemma 5 *For any $n \in \mathbb{N}$, $n \geq 6$, the quotient* $\frac{\binom{n-1}{5}}{n-5}$

(a) is in $\mathbb{N}$, if n is not a multiple of 5,

(b) equals $125\binom{\ell+1}{4} + 25\binom{\ell}{2} + \frac{1}{5}$, if $n = 5\ell$.

Lemma 6 *The recurrence relation*

$$T(n) = \left\lceil \frac{n}{n-5} T(n-1) \right\rceil \text{ for } n > 7 \text{ and } T(7) = 29,$$

has solution

$$T(n) = 2\left\lfloor \frac{\binom{n}{5}}{5} \right\rfloor + \binom{n}{5} - 2\left\lfloor \frac{n}{25} \right\rfloor .$$

4 The moment curve and an upper bound

Lemma 7 *Let S be a set of n points on the moment curve $\gamma(t) = (t, t^2, t^3)$, with $t > 0$. Then, all five-tuples of S are of Type B, and $\sum_{k=0}^{n-4} k \cdot s_k = 2\binom{n}{5}$.*

The proof is based on point-in-sphere determinant tests.

Corollary 1 *Among all sets S of n points in convex and general position in $\mathbb{R}^3$, points on the moment curve minimize $\sum_{k=0}^{n-4} k \cdot s_k$.*

Next, we will use the inversion transformation to construct point sets of arbitrary size in $\mathbb{R}^3$ with all of its five-tuples being Type C. The inversion is determined by two parameters: The center of inversion O and the radius of inversion R. Two points p and p' in $\mathbb{R}^3$ are said to be inverses of each other if:

1. The points p and p' lie in the same half-line with origin in O.
2. The Euclidean distances $|\overline{Op}|$ and $|\overline{Op'}|$ in $\mathbb{R}^3$ satisfy $R^2 = |\overline{Op}||\overline{Op'}|$.

The following is a well-known inversion's property which is key to construct such sets.

Property 1 *The inverse of any sphere $\bigcirc$ that does not pass through the center of inversion is a sphere $\bigcirc'$ that also does not pass through the center of inversion. Also, if the center of inversion is in the interior of $\bigcirc$, then the interior of $\bigcirc$ transforms to the exterior of $\bigcirc'$ and the exterior of $\bigcirc$ transforms into the interior of $\bigcirc'$.*

We show that all the spheres defined by four points on the moment curve $\gamma(t)$ with $0 < t < \frac{1}{10}$ intersect, in point $O = \left(0, \frac{1}{2}, 0\right)$. Then, inversion with center O and radius $R = 1$ transforms each five-tuple of Type B of points from $\gamma(t)$ with $0 < t < \frac{1}{10}$ into one of Type C. Thus:

Theorem 8 *Let S be a set of n points in general position in $\mathbb{R}^3$. Then, $\sum_{k=0}^{n-4} k \cdot s_k \leq 3\binom{n}{5}$ and the bound is tight.*

Corollary 2 *Among all sets of n points in general position in $\mathbb{R}^3$, $\sum_{k=0}^{n-4} k \cdot s_k$ is maximized for a set S of n points on the curve $\delta(t)$ given by*

$$\left(\frac{4t}{4t^6 + 4t^4 + 1}, \frac{4t^6 + 4t^4 + 8t^2 - 3}{8t^6 + 8t^4 + 2}, \frac{4t^3}{4t^6 + 4t^4 + 1} \right)$$

with $0 < t < \frac{1}{10}$. All the five-tuples of S are of Type C.

The curve $\delta(t)$ is the inversion of the moment curve used in Theorem 8.

5 A universal constant for points in spheres containment

Let S be a set of $n \geq d+2$ points in general position in $\mathbb{R}^d$. Let $P_d(S)$ be the probability that the sphere passing through $d+1$ points chosen uniformly at random from S, contains another point chosen uniformly at random from the remaining points of S. We have

$$P_d(S) = \frac{\sum_{k=0}^{n-d-1} k \cdot s_k}{(d+2)\binom{n}{d+2}}. \qquad (4)$$

To see this, first observe that there are $\binom{n}{d+2}$ ways to choose $d+2$ different points from S, and among them, any can be the point to test to be inside or outside the sphere determined by the other $d+1$ points. On the other hand, for a sphere enclosing k points of S, we count k times a sphere containing another point. Altogether, there are $\sum_{k=0}^{n-d-1} k \cdot s_k$ spheres determined by $d+1$ points that contain another point from S. All these spheres containing a point are equally likely to be chosen.

We define $P_d(n) = \min P_d(S)$, where the minimum is taken among all sets S of n points in general position in $\mathbb{R}^d$, and $P_d^* = \lim_{n\to\infty} P_d(n)$. We show that this limit exists.

Lemma 9 *For each dimension $d \geq 2$, there exists a constant $0 \leq P_d^* \leq 1$ such that*

$$P_d^* = \lim_{n\to\infty} P_d(n).$$

From Theorem 3 and the proof of Lemma 9 we obtain the following corollary:

Corollary 3

$$P_3^* \geq \frac{7}{25}.$$

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