Factorization and inversion of coupler curves

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Abstract

We provide a general framework for deciding “how far away from a factorization” a given polynomial is, and apply an implementation in `julia` to the case of coupler curves of 4-bar linkages. We also clarify when the inversion of the coupler curve in a particular circle is a conic, and classify the conics arising in this way.

1 Introduction

Perhaps no mechanical linkage is more recognizable than the four-bar linkage. Its four rigid bars, called links, are connected to one another by revolute joints, and their movement has one degree of freedom in the non-degenerate case. Of the four links, the coupler link ℓ_3 is the only one not rigidly connected to the stationary ground link ℓ_1. It is usually rigidly connected to a coupler triangle Δ. The trace point T, the third vertex of Δ, traces the coupler curve κ as the linkage moves through its possible positions, cf. Figure 1.

Antiparallelograms consist of two sets of opposite links of equal length in which the longer links cross. Their coupler curves are bicircular quartics, with highest-order term $x^4 + 2x^2y^2 + y^4$. Playing with these quartics suprised us with the observation that applying a circular inversion in the correct place turns them into conic sections! Why should this be?

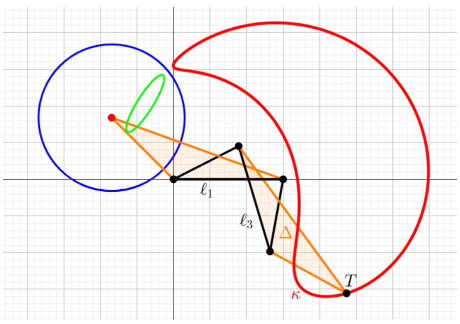

Figure 1: An antiparallelogram linkage in **black**, the link on the x axis being “grounded”. In **red** is a coupler curve traced by a point on the shaded coupler triangle, in **blue** a circle on a similar triangle. Inversion of the red curve in the blue circle yields the **green** ellipse.

For more general four-bar linkages this is no longer true, but many things are similar. The general coupler curve is now a tricircular sextic [3], with highest-order terms $(x^2 + y^2)^3$. These curves can achieve a much wider range of shapes, but in the vicinity of special cases they “almost” decompose into two or more constituent curves, cf. Figures 2 and 3.

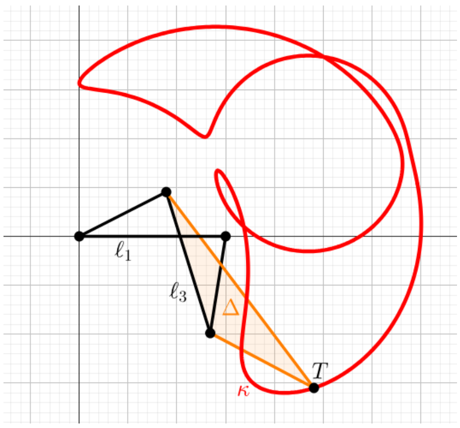

Figure 2: The **red** coupler curve of a general four-bar linkage that is very close to an antiparallelogram almost decomposes into the product of a circle and the bicircular quartic from Figure 1.

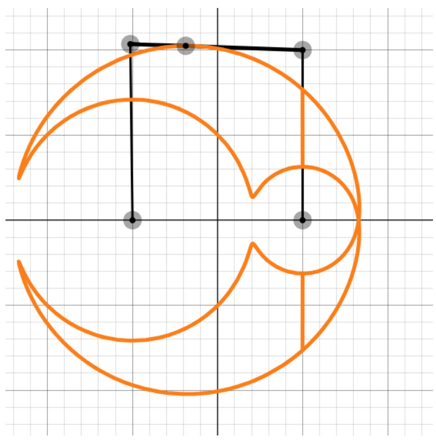

Figure 3: The curve of a general four-bar linkage that is very close to a square almost decomposes into the product of three circles.

Apart from solving the mystery of the inversions, the goal of this paper is to take the first step towards a Fourier-like representation of k-circular planar curves in terms of a list of "almost factors" and a minimal error term that needs to be added to achieve factorization. For this, we take a resolutely geometric view:

2 The geometry of factorization

How far is, say, a quadratic polynomial

$$ax^2 + bxy + cy^2 + dx + ey + f \tag{2.1}$$

in two variables away from decomposing into a product $(a_x x + a_y y + a_0)(b_x x + b_y y + b_0)$ of linear forms?

To answer this geometrically, we combine the coefficients of the linear forms into the matrix

$$\begin{bmatrix} a_x \\ a_y \\ a_0 \end{bmatrix} \begin{bmatrix} b_x & b_y & b_0 \end{bmatrix} = \begin{bmatrix} a_x b_x & a_x b_y & a_x b_0 \\ a_y b_x & a_y b_y & a_y b_0 \\ a_0 b_x & a_0 b_y & a_0 b_0 \end{bmatrix}$$

and rearrange this 3×3 Segre embedding $S_{(1,1);3}$ of two linear forms in 3 variables by reading it row-wise into a column vector $s = s_{(1,1);3}$ of length 9. This column vector, in turn, projects to the Veronese embedding of the quadratic form via the matrix

$$P = P_{(1,1);3} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The entries of Ps are the coefficients of the monomials $(x^2 : xy : y^2 : x : y : 1)$ in the product. For instance, the second entry is $a_x b_y + a_y b_x$, the coefficient of xy.

The image in $\mathbb{R}^6$ under P thus parametrizes the quadratic forms that split into linear factors. Eliminating the variables $a_x, a_y, a_0, b_x, b_y, b_0$ from the equations $a = a_x b_x$, $b = a_x b_y + a_y b_x$, ..., $f = a_0 b_0$ using polynomial implicitization [2, Theorem 3.3.1] shows that the coefficients of factorizable forms (2.1) satisfy

$$ae^2 + cd^2 + fb^2 - 4acf - bde \;=\; 0, \tag{2.2}$$

and we are tempted to call it a day — do we not now have a precise criterion for when a quadratic polynomial splits? For instance, $(x+1)(x-1) = x^2 - 1$ splits because the only non-zero coefficients are $a = 1$, $f = -1$, and they satisfy (2.2), as do the coefficients of $x^2 + 2xy + y^2$, for example.

But not so fast: According to (2.2), the polynomial $x^2 + y^2$, whose only non-zero coefficients are $a = c = 1$ should also split — and indeed it does, into $(x + \boldsymbol{i}y)(x - \boldsymbol{i}y)$! Alas, algebraic geometry works best over the complex numbers, and in order to gain insight about real coupler curves we need to work a bit more.

We obtain the preimage under P of the coefficients $q = (a, b, \dots, f) \in \mathbb{R}^6$ of a general quadratic polynomial by solving the non-homogeneous linear system $Px = q$. Written as a 3×3 matrix, it turns out to be

$$\begin{bmatrix} a & b - \lambda_1 & d - \lambda_2 \\ \lambda_1 & c & e - \lambda_3 \\ \lambda_2 & \lambda_3 & f \end{bmatrix} \qquad \text{for } \lambda_i \in \mathbb{R}, \tag{2.3}$$

and such a point in $\mathbb{R}^9 \cong \mathbb{R}^{3\times 3}$ in the preimage of q under P corresponds to a product of linear factors precisely if it has rank 1, i.e., its 2×2 minors vanish.

Some very interesting vanishing minors say that

$$\lambda_1^2 = b\lambda_1 - ac, \quad \lambda_2^2 = d\lambda_2 - af, \quad \lambda_3^2 = e\lambda_3 - cf. \tag{2.4}$$

They tell us that for the factorization to be real, we need that $b\lambda_1 \geq ac$, $d\lambda_2 \geq af$, $e\lambda_3 \geq cf$.

Moreover, our geometric setup allow us to answer the question of *how far away* a polynomial q is from factoring. One way would be to determine the minimum distance in $\mathbb{R}^9$ of a point in the preimage of q under P to the variety of rank 1 matrices, but that does not seem to carry a lot of information.

More intrinsically, we decide to measure the *minimum total degree of a polynomial perturbation* $\varepsilon = \varepsilon_6 x^2 + \varepsilon_5 xy + \cdots + \varepsilon_1$ that must be added to q such that $q + \varepsilon$ factors, i.e., some matrix in the preimage of $q + \varepsilon$ has rank 1. This makes sense because total degree is preserved by affine changes of variables, but individual terms are not. Additionally, we of course would like ε to be small, for instance in the sense that $\|(\varepsilon_6, \varepsilon_5, \dots, \varepsilon_1)\|_2^2$ be as small as possible.

For example, allowing perturbation by linear polynomials means looking for rank 1 matrices of the form

$$S_{(1,1);1}(q, \lambda, \varepsilon) = \begin{bmatrix} a & b - \lambda_1 & d + \varepsilon_3 - \lambda_2 \\ \lambda_1 & c & e + \varepsilon_2 - \lambda_3 \\ \lambda_2 & \lambda_3 & f + \varepsilon_1 \end{bmatrix},$$

where $\lambda_i, \varepsilon_i \in \mathbb{R}$ and additionally

$$\begin{aligned} \lambda_1^2 \;&=\; b\lambda_1 - ac \;\geq\; 0, \\ \lambda_2^2 \;&=\; (d + \varepsilon_3)\lambda_2 - a(f + \varepsilon_1) \;\geq\; 0, \\ \lambda_3^2 \;&=\; (e + \varepsilon_2)\lambda_3 - c(f + \varepsilon_1) \;\geq\; 0. \end{aligned}$$

To generalize this to factorizing any polynomial of degree $d + e$ in a fixed number n of variables into two factors of degrees d, e, up to a perturbation of degree δ, set $\mu(D) := \binom{D+n-1}{D}$, the dimension of the Veronese embedding of a polynomial of degree D in n variables. We also set $\mu(-1) := 0$ and accord that the zero polynomial has degree -1.

The matrix $P_{(d,e);n}$ then has size $\mu(d+e) \times \mu(d)\mu(e)$ and kernel of dimension $k := k(d, e, n) := \mu(d)\mu(e) - \mu(d + e)$. Inside the preimage $S_{(d,e);\delta}(q, \lambda, \varepsilon)$ of size $\mu(d) \times \mu(e)$ we find entries $\lambda_1, \dots, \lambda_k$ and $\varepsilon_1, \dots, \varepsilon_{\mu(\delta)}$.

Theorem 2.1 *For integers $d, e, n \geq 1$, deciding whether a real polynomial q of degree $d+e$ in n variables, perturbed by a polynomial of degree $-1 \leq \delta \leq d+e$, splits into real factors of degrees d, e amounts to solving the quadratic optimization problem* $\mathrm{QOPT} = \mathrm{QOPT}(d, e, n, \delta, q)$ *defined as*

$$\begin{aligned} \min \quad & 1 + \sum_{i=1}^{\mu(\delta)} \varepsilon_i^2 \\ \text{s.t.} \quad m &= 0, \qquad m \text{ a } 2 \times 2 \text{ minor of } S_{d+e,\delta}(q, \lambda, \varepsilon), \\ r_i &\geq 0, \qquad i = 1, \ldots, k(d,e,n) \end{aligned}$$

where $m = \lambda_i^2 - r_i$ are the 2×2 minors in which λ_i is the only λ-variable that occurs.

We have implemented [5] this quadratic optimization problem in `julia 1.11.4` [4], using the backend connecting it to the Gurobi 12.0.1 optimizer. To see it in action, consider the parametrized family of coupler curves

$$\begin{aligned} &(x^2+y^2)^2 - (2t^2 - 4t + \tfrac{7}{2})x^2 - (2t^2 - 4t - \tfrac{7}{2})y^2 \\ &-x - 6(t+1)y + t^4 - 4t^3 + \tfrac{19}{2}t^2 - 11t + \tfrac{69}{16} \;=\; 0, \end{aligned} \tag{2.5}$$

whose parameter t encodes the position of the trace point of a 4-bar linkage in which the ground and coupler links have length 2, and the others have length 1.

This linkage has two types of configurations, a parallelogram and an antiparallelogram, which meet when the entire linkage is splayed out on a line, cf. Figure 4.

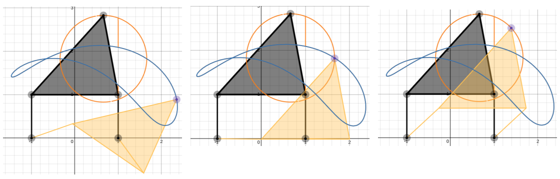

Figure 4: Three types of configuration of a certain 4-bar linkage, shown "at rest" (dark) and "in motion" (light). *Left:* the trace point in the antiparallelogram configuration traces a quartic. *Middle:* the linkage degenerates into a straight-line configuration. *Right:* The parallelogram configuration moves in a circle.

In consequence, the tricircular sextic that describes the motion of the coupler point factors into a conic, which by tricircularity must be a circle, and a family of quartics parametrized by t.

Running our factorization algorithm on this family for $0 \leq t \leq 0.2$ yields that in all cases, the degree 4 coupler curve can be factored into a product of conics after allowing for a linear error term, cf. Figure 5.

3 Inversions of coupler curves

Let b denote the symmetry axis of the antiparallelogram linkage of Figures 1 and 2, and let γ be the envelope of all the symmetry axes obtained as the

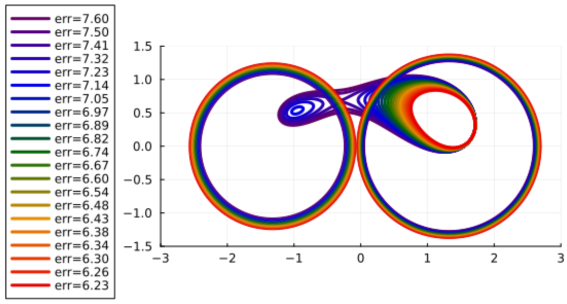

Figure 5: Almost-factorizing a 1-parameter family of quartic curves (fat) into two circles (thin). The colors represent values of the parameter t in equation 2.5 , from 0 **(red)** to 0.2 **(purple)**. The given error is the Euclidean 2-norm of the affine term that needs to be added to achieve factorization.

linkage moves, cf. [2, Definition 3.4.5]. Due to its particular properties when interpreted as a gear [1, Section 3.3], we call γ the gear conic of the antiparallelogram. Moreover, let F be the reflection of the trace point E around b.

Theorem 3.1 *(a) The gear curve γ of an antiparallelogram is a conic, (b) as is the inversion $i(\kappa)$ of the coupler curve κ in the unit circle around F. (c) When F lies on the gear conic γ, the inversion $i(\kappa)$ is a parabola (Figure 6). If F and the foci of γ are separated by γ, $i(\kappa)$ is a hyperbola (Figure 7), otherwise, an ellipse (Figure 1).*

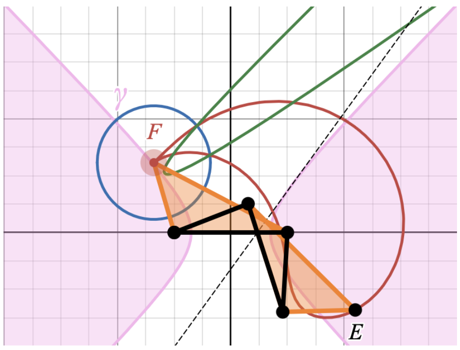

Figure 6: An antiparallelogram linkage like the one in Figure 1, but with its coupler curve adjusted so that F lies on the **pink** gear conic. As per Theorem 3.1, the **green** inverted coupler curve is a parabola.

To close off the paper, we have just seen that the inversions of coupler curves of antiparallelogram linkages turn into conics under inversion in a circle. What other curves share this behavior? And which conics arise in this way?

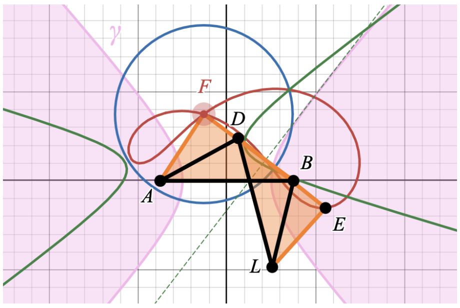

Figure 7: The same construction as in Figure 6, but with the triangles altered such that the inversion center is outside the pink gear conic. This makes the **green** inverted coupler curve a hyperbola.

Theorem 3.2 *Let $f \in \mathbb{R}[x, y]$ be a polynomial of degree d, and assume that $r^2 := x^2 + y^2$ does not divide f. Then the inversion of the vanishing locus of f is a conic or a line iff f has either of the forms*

$$f = a + b_x x + b_y y + cr^2, \quad (3.1)$$
$$f = g^{(2)}(x, y) + r^2\big(b_x x + b_y y\big) + c\, r^4, \quad (3.2)$$

where $g^{(2)}(x, y)$ is a homogeneous *polynomial of degree* 2, *and $a, b_x, b_y, c \in \mathbb{R}$ are constants.*

Observation 3.3 *Note that the degrees of the homogeneous parts of f are contained in either the set $\{0, 1, 2\}$, if f is of the form (3.1), or in the set $\{2, 3, 4\}$, if f is of the form (3.2). Thus, in order for its inversion to be a conic, a polynomial must be either a conic or a quartic with a double point at the origin and certain restrictions on its coefficients.*

Theorem 3.4 *The conics of the form (2.1) representable as inversions of the coupler curve of an antiparallelogram linkage family $\mathcal{F}$ are the ones whose coefficients satisfy $2b = de$, $a < d^2/4$, and $c > (d^2/4 - a) + e^2/4$.*

4 Open questions and future work

(1) Experimentation suggest that the optimal circles that almost-factor the coupler curve of the 4-bar linkage are centered either on the anchor points of the ground link, A and B in Figure 7, or on the final anchor of a four-bar cognate, i.e., a linkage with different dimensions that traces the exact same coupler curve, cf. [6]. Mathematically, this anchor lies where the tracer point lands if the coupler triangle is shifted and scaled to lie on the ground link. Computing examples as in Figure 5 confirms this result, although the authors are very puzzled as to why this should be the case.

(2) Is there some operation that simplifies coupler curves of general four-bar linkages? For antiparallelograms, inverting in a circle yields conics. Does this generalize, and how?

(3) It is not clear to the authors that the squared Euclidean 2-norm $\sum_{i=1}^{\mu(\delta)} \varepsilon_i^2$ of the perturbation terms that must be added to q in order to permit factorization is really a good choice of objective function for QOPT, as these coefficients perhaps carry intrinsic geometric information. What better substitutes could there be?

(4) The code in [5] currently only treats the Segre embedding of two factors. This could of course be extended to more factors in order to directly treat the almost-factorization into three circles of Figure 3.

Acknowledgements

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A The geometry of factorization

For any degree $D \in \mathbb{N}$, we write $M_D = M_{D,n}$ for the set of $\mu(D) = \binom{D+n-1}{D}$ exponent vectors of monomials of degree d in n variables, ordered lexicographically. We use these sets of exponent vectors of monomials to label the coordinates of the ambient spaces of the Segre and Veronese embeddings:

- The Segre embedding $\Sigma_{(d,e);n}$ of $\mathbb{P}^{\mu(d)-1} \times \mathbb{P}^{\mu(e)-1}$ parametrizes pairs of polynomials of respective degrees d and e. We label the coordinates of its ambient space $\mathbb{P}^{\mu(d)\mu(e)-1} \supset \Sigma_{(d,e);n}$ by $M_d \times M_e$.
- The Veronese embedding $V_{d+e,n}$ parametrizes polynomials of degree $d+e$. We label the coordinates of its ambient space $\mathbb{P}^{\mu(d+e)-1} \supset V_{d+e,n}$ by M_{d+e}.

Proposition A.1 *For $n, d, e \in \mathbb{N}$, the $\mu(d+e) \times \mu(d)\mu(e)$ matrix $P_{(d,e);n}$ that expresses the linear projection from $\Sigma_{(d,e);n}$ to $V_{d+e,n}$ in these coordinates is zero everywhere, except for a 1 in*

- *the row corresponding to the coordinate labeled by the sum $m + m'$ and*
- *the column corresponding to the coordinate labeled by the concatenation (m, m')*

for all $m \in M_d$ and $m' \in M_e$. Each column of $P_{(d,e);n}$ contains exactly one entry 1.

Proof. The exponent vectors $m \in M_d$ and $m' \in M_e$ contributing to any fixed exponent vector $m'' \in M_{d+e}$ are precisely the ones such that $m + m' = m''$, and each concatenation (m, m') occurs exactly once. □

Definition A.2 *For $n, d, e \in \mathbb{N}$, iteratively construct the columns of a matrix $K_{(d,e);n}$ as follows:*
(1) *Start with an empty matrix of $\mu(d)\mu(e)$ rows.*
(2) *For each row $R_{m''}$ of $P_{(d,e);n}$, labeled by an exponent vector $m'' \in M_{d+e}$, let σ be the number of 1s contained in it. This is just the number of ways of obtaining m'' by summing two exponent vectors $m \in M_d$, $m' \in M_e$.*
(3) *If $\sigma = 1$, do nothing.*
(4) *Otherwise, write down the σ many pairs $(\Pi_1, \ldots, \Pi_\sigma) = \big((m, m') \in M_d \times M_e : m + m' = m''\big)$ in some order, and for $\ell = 2, \ldots, \sigma$, append a column vector to $K_{(d,e);n}$ that has a 1 in the row labeled by Π_1, and a -1 in the row labeled by Π_ℓ.*

Proposition A.3 *The matrix $K_{(d,e);n}$ has $\mu(d)\mu(e)$ rows and $\mu(d)\mu(e) - \mu(d+e)$ columns, and its columns form a basis of $\ker P_{(d,e);n} : \Sigma_{(d,e);n} \to V_{d+e,n}$.*

Proof. The stated columns obviously lie in the kernel of $P_{(d,e);n}$ and are linearly independent. Counting them confirms that they generate the entire kernel. □

Observe that there is one λ variable attached to each generator of $\ker P_{(d,e);n}$, and that each column of $K_{(d,e);n}$ has exactly one "+1" and one "−1" entry by Definition A.2 (4). We conclude that each variable λ_i occurs in exactly two entries of $S_{(d,e);n}(q, \lambda, \varepsilon)$, and these entries give rise to the special 2×2 minors $\lambda_i^2 - r_i$ in Theorem 2.1 that contribute the non-negativity conditions insuring realness of the solution.

As in the degree 2 case discussed in the main part of the paper, asking that the degree-$(d+e)$ polynomial q split into factors of degrees d and e amounts to asking that the matrix $S_{(d,e);n}$ lie in the image of the Segre embedding of $\mathbb{RP}^{\mu(d)-1} \times \mathbb{RP}^{\mu(e)-1}$, i.e., that it have rank 1; and this in turn is ensured by the vanishing of the 2×2 minors.

This concludes the proof of the structural part of Theorem 2.1. The choice of objective function is in some sense arbitrary, and different choices are possible; we choose to minimize the (square of) the Euclidean 2-norm of the perturbation, and add a constant summand +1 for greater numerical stability.

B Inversion of coupler curves

B.1 Proof of Theorem 3.1 (a)

We coordinatize the revolute joints of the antiparallelogram linkage as follows: $A = (a, b)$, $B = (a+d, b)$,

$$D \;=\; A + \ell \cdot \left(\frac{t^2-4}{t^2+4}, \frac{4t}{t^2+4}\right),$$

and set $H = (B+D)/2$. The coordinates for D come from rationally parametrizing the motion of the crank with $t \in \mathbb{R}$. The axis of symmetry b is the line orthogonal to $B - D$ through H, and this allows us to calculate the equation $G(x, y, t) = 0$ of the motion of the crank. We obtain the envelope [2, Definition 3.4.5] of this parametric family of curves by eliminating t from the system of equations $G(x, y, t) = \partial G(x, y, t)/\partial t = 0$:

$$4(d^2-\ell^2)(x-a-d/2)^2 - 4\ell^2(y-b)^2 \;=\; \ell^2(d^2-\ell^2). \quad \text{(B.1)}$$

The gear curve is thus a hyperbola for $d^2 > \ell^2$, the double line $y = b$ for $d^2 = \ell^2$, and an ellipse for $d^2 < \ell^2$.

B.2 Proof of Theorem 3.1 (b)

We need some concepts from classical algebraic geometry. Throughout, C denotes an affine algebraic plane curve, T a line tangent to C, E an arbitrary point in the plane, and $\mathcal{T}$ the set of all tangent lines to C.

Definition B.1 *The foot point $f_T(E)$ of E with respect to T is the closest point to E on T. The pedal curve $\pi_E(C)$ of C with respect to E is $\pi_E(C) = \{f_T(E) : T \in \mathcal{T}\}$. The orthotomic curve $o_E(C)$ of C with respect to E is the trace of all reflections of E across T, as $T \in \mathcal{T}$.*

Observation B.2 *Because reflecting a point in a line means going twice as far as towards the foot point, the orthotomic curve $o_E(C)$ is just $\pi_E(C)$ scaled by a factor of 2 from the point E.*

Now fix a quadric Q in the plane, which will usually be a circumference of radius 1 centered at E. It induces a polarity, classically called a reciprocation, between points and lines in the plane.

Definition B.3 *The polar line $p^\perp$ of a point p with respect to Q is the line through the points on Q whose tangents pass through P. The polar point or pole $H^\perp$ of a line H is the intersection of the tangent lines to Q at the intersections of Q with H.*

Thus, the operations $p \mapsto p^\perp =: H$ and $H \mapsto H^\perp$ are inverses of each other. See [A1] for a more extensive discussion and the extension to higher dimensions.

Definition B.4 *The polar reciprocal curve $\rho_Q(C)$ of C w.r.t Q is $\rho_Q(C) = \{T^\perp : T \in \mathcal{T}\}$.*

Proposition B.5 (The Dual Conic Theorem) (cf. [A2, Section 9.3]) *If Q, Q' are conics, then $\rho_Q(Q')$ is also a conic.*

Here is a much older definition for the special case where Q is the circumference of radius r centered at E:

Definition B.6 *(cf. [A3, §VII.233]) The polar reciprocal curve $\rho_{E,r}(C)$ is the locus of points Z such that $|\overline{E\, f_T(E)}| \cdot |\overline{EZ}| = r^2$, where $T \in \mathcal{T}$.*

The advantage of this older definition is that it immediately makes clear the following result:

Proposition B.7 *Let Q be the circumference of radius 1 centered at E. Then the polar reciprocal curve $\rho_{E,1}(C)$ is the inversion with respect to Q of the pedal curve $\pi_E(C)$.* □

We can now prove Theorem 3.1 (b):

Proof. Let $C = \gamma(\mathcal{F}) = \gamma$ be the gear curve of the family of antiparallelograms, which is a conic by Theorem 3.1 (a). The point F traces the orthotomic $o_E(\gamma)$, which by Observation B.2 is a copy of the pedal curve $\pi_E(\gamma)$, scaled from E by a factor of 2. The point F' is the inversion of F around the unit circle centered at E, so it traces $\frac{1}{2}$ times the inversion of the pedal curve $\pi_E(\gamma)$. By Proposition B.7, $\pi_E(\gamma)$ is the inverse of the polar reciprocal curve $\rho_{E,1}(\gamma)$, whence F' traces $\frac{1}{2}\rho_{E,1}(\gamma)$, where the scaling happens from E — and this trace, by definition, is just the inverted trace $\sigma(\mathcal{F})$. But by Proposition B.5, the polar reciprocal $\rho_{E,1}(\gamma)$ of the conic γ with respect to the unit circle centered at E is a conic, and we conclude that the inverted trace $\sigma(\mathcal{F}) = \frac{1}{2}\rho_{E,1}(\gamma)$ is also a conic. □

B.3 Proof of theorem 3.1 (c)

When the coupler curve is inverted, any point coinciding with the center of inversion gets mapped to infinity. Since the center of inversion F and the trace point E are mirrors about the line of symmetry, and as the gear conic γ is the envelope of this line, when F is inside of γ it will not lie on any line of symmetry. As such, E will always be distinct from F, so no point on the coupler curve will be sent to infinity. The only conic without points at infinity is the ellipse, cf. Figure 1. If F is on γ, it will lie on the line of symmetry exactly once, so there will be exactly one point at infinity in the resulting conic, yielding a parabola, cf. Figure 6. Finally, if F is outside of γ, it will lie on two distinct lines of symmetry, resulting in two points at infinity, so the inverted coupler curve will be a hyperbola, cf. Figure 7.

Definition B.8 *The strict transform (under inversion in the unit circumference) of a polynomial $f(x, y)$ is the polynomial $r^{2m} f(x/r^2, y/r^2)$ obtained from the rational function $f(x/r^2, y/r^2)$ by multiplying it with the smallest power of r^2 that removes the denominator.*

B.4 Proof of Theorem 3.2

In addition to the statement, we will additionally show that the strict transforms of f in cases (3.1) and (3.2) are, respectively,

$$r^2\overline{f} \;=\; a\, r^2 + b_x\, x + b_y\, y + c, \tag{B.2}$$

$$r^4\overline{f} \;=\; g^{(2)}(x, y) + b_x\, x + b_y\, y + c, \tag{B.3}$$

where $\overline{f} = \sum_{i\geq 0} f_i/r^{2i}$ is the inversion of f.

To see all this, let $f = \sum_{e=0}^{d} f_e(x, y) \in \mathbb{R}[x, y]$ be a polynomial of degree d with homogeneous parts

$$f_e(x, y) \;=\; \sum_{i+j=e} a_{ij}\, x^i y^j,$$

and assume that $r^2 := x^2 + y^2$ does not divide f. Note that the inversion of f is

$$\overline{f} \;=\; \sum_{i\geq 0} \frac{f_i}{r^{2i}},$$

where almost all terms in this expression are zero. If $f_i \neq 0$, we factor $f_i = r^{2k_i} f'_i$ by dividing out the highest possible power of r^2. Then $\deg f'_i = i - 2k_i \geq 0$, so that k_i is constrained by

$$0 \;\leq\; k_i \;\leq\; i/2. \tag{B.4}$$

This factorization yields

$$\overline{f} \;=\; \sum_{i\geq 0} \frac{f'_i}{r^{2i-2k_i}},$$

and we multiply this expression by the largest power

$$2m \;:=\; \max\big\{2i - 2k_i : i \geq 0, f_i \neq 0\big\} \qquad \text{(B.5)}$$

of a denominator, to obtain

$$r^{2m}\overline{f} \;=\; \sum_{i\geq 0} f_i'\, r^{2(m+k_i-i)}.$$

The i-th term in this polynomial is of degree

$$(i-2k_i) + 2(m+k_i-i) \;=\; 2m-i,$$

so in order for $r^{2m}\overline{f}$ to be a conic, i.e., $2m - i \in \{0,1,2\}$, the only non-zero terms must be the ones with $i \in \{2m-2, 2m-1, 2m\}$. Substituting these values into (B.5) and dividing by 2 shows that m is

$$\max\big\{2m-2-k_{2m-2}, 2m-1-k_{2m-1}, 2m-k_{2m}\big\}. \quad \text{(B.6)}$$

We now distinguish cases according to which term in (B.6) achieves the maximum:

(1) If the maximum is achieved by the first term, $m = 2m-2-k_{2m-2}$, then

- $k_{2m-2} = m-2$;
- $m \geq 2m-1-k_{2m-1}$, i.e., $k_{2m-1} \geq m-1$. Because also $k_{2m-1} \leq m - \frac{1}{2}$ by (B.4), we conclude that $k_{2m-1} = m-1$.
- $m \geq 2m - k_{2m}$, i.e., $k_{2m} \geq m$. Using (B.4) again, we arrive at

$$\big(k_{2m-2}, k_{2m-1}, k_{2m}\big) \;=\; \big(m-2, m-1, m\big).$$

(2) If the maximum is achieved by the second term, $m = 2m-1-k_{2m-1}$, then

- $k_{2m-1} = m-1$;
- $m \geq 2m-2-k_{2m-2}$, i.e., $m-1 \geq k_{2m-2} \geq m-2$ using (B.4);
- $m \geq 2m-k_{2m}$, which as in case (1) again yields $k_{2m} = m$. We conclude that

$$\big(k_{2m-2}, k_{2m-1}, k_{2m}\big) \;=\; \big(m-1-\delta, m-1, m\big),$$

for some $\delta \in \{0,1\}$.

(3) Finally, if the maximum is achieved by the third term, then $m = 2m - k_{2m}$, so that

- $k_{2m} = m$;
- $m \geq 2m-2-k_{2m-2}$ so that again $k_{2m-2} = m-1-\delta$ as in case (2);
- $m \geq 2m-1-k_{2m-1}$, so that again $k_{2m-1} = m-1$ as in case (1), and therefore

$$\big(k_{2m-2}, k_{2m-1}, k_{2m}\big) \;=\; \big(m-1-\delta, m-1, m\big),$$

with $\delta \in \{0,1\}$.

These results yield the following table:

i	$2m-2$	$2m-1$	$2m$
k_i	$m-1-\delta$	$m-1$	m
$\deg f_i' = i - 2k_i$	$2\delta \in \{0,2\}$	1	0
exponent $2(m+k_i-i)$ of r	$2-2\delta \in \{2,0\}$	0	0

With this data, the polynomials f whose inversion is a conic and their strict transforms are

$$\begin{aligned} f &= \sum_{i\geq 0} r^{2k_i} f_i' \\ &= r^{2(m-1-\delta)}(f_{2m-2}')^{(2\delta)} + r^{2(m-1)}(f_{2m-1}')^{(1)} + r^{2m}(f_{2m}')^{(0)} \\ r^{2m}\overline{f} &= \sum_{i\geq 0} r^{2(m+k_i-i)} f_i' \\ &= r^{2-2\delta}(f_{2m-2}')^{(2\delta)} + (f_{2m-1}')^{(1)} + (f_{2m}')^{(0)}. \end{aligned}$$

Writing

$$\begin{aligned} (f_{2m-2}')^{(2\delta)} &= \begin{cases} a & \text{if } \delta = 0 \\ g^{(2)}(x,y) & \text{if } \delta = 1, \end{cases} \\ (f_{2m-1}')^{(1)} &= b_x\, x + b_y\, y, \\ (f_{2m}')^{(0)} &= c, \end{aligned}$$

where $g^{(2)}(x,y)$ is a *homogeneous* polynomial of degree 2, this expresses f as

$$\begin{cases} r^{2m-2}\big(a + b_x\, x + b_y\, y\big) + c\, r^{2m} & \text{if } \delta = 0 \\ r^{2m-4} g^{(2)}(x,y) + r^{2m-2}\big(b_x\, x + b_y\, y\big) + c\, r^{2m} & \text{if } \delta = 1 \end{cases}$$

or

$$\begin{cases} r^{2m-2}\big(a + b_x\, x + b_y\, y + cr^2\big) & \text{if } \delta = 0 \\ r^{2m-4}\Big(g^{(2)}(x,y) + r^2\big(b_x\, x + b_y\, y\big) + c\, r^4\Big) & \text{if } \delta = 1, \end{cases}$$

with strict transforms

$$r^{2m}\overline{f} \;=\; \begin{cases} a\, r^2 + b_x\, x + b_y\, y + c & \text{if } \delta = 0 \\ g^{(2)}(x,y) + b_x\, x + b_y\, y + c & \text{if } \delta = 1. \end{cases}$$

Supposing additionally that r^2 does not divide f yields $m = 1$, resp. $m = 2$, and the claim.

C Bibliography for the Appendix

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