Preservation of Euclideaness in oriented matroids and applications

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1 Introduction

These results are mainly from the preprint [9] from 2025 and were already presented at EuroCG2025 in a geometrical way. We add here some more results and use the more formal language of oriented matroids. For the details of oriented matroid (programs), we refer to [1], Chapters 3,4 and 10. An *oriented matroid program* is a triple $(\mathcal{O}, g, f)$ where $\mathcal{O}$ is an oriented matroid with groundset $E_n \dot\cup \{f, g\}$ such that $f \neq g$ are not loops or coloops. Two cocircuits X and Y of an oriented matroid are *neighbours* iff they are *conformal* — meaning that $sep(X, Y) \neq 0$, i.e., no elements separate them — and *comodular*, which means that the zeroset of $X \circ Y$ has corank 2 in the underlying matroid. The cocircuit graph G_f of an oriented matroid program has as vertices the cocircuits of $\mathcal{O} \setminus f$ with $g = +$, where edges connect cocircuits that are neighbours in $\mathcal{O} \setminus f$.

We briefly describe the geometrical intuition behind this situation in the case where the oriented matroid is *realizable*. In this case, the elements other than f and g correspond to an arrangement of affine hyperplanes, the element g represents the hyperplane at infinity, and f represents an objective function. (In a homogeneous representation all these things correspond to vectors in $\mathbb{R}^d$: the affine hyperplane $ax = b$ is represented as a vector (a, b), the hyperplane at infinity as the vector $(0, 1)$, and the objective function as a vector $(v, *)$ where v is its gradient and * is an arbitrary real number, irrelevant for the rest (it can be taken to be 1). Then cocircuits correspond to vertices of the affine hyperplane arrangement, and two of them are neighbours if and only if they are adjacent in the 1-skeleton of the arrangement. In fact, the 1-skeleton of the arrangement is the "cocircuit graph" referred in the rest of the paper. In general, oriented matroids correspond to arrangements of *pseudospheres*, where a pseudosphere of rank $d-1$ is a subset of the standard sphere S^d homeomorphic to S^{d-1} (see [3]). An oriented matroid is realizable, if and only if its pseudosphere arrangement is "stretchable" — that is, there exists a hypersphere arrangement (or a homogenized affine hyperplane arrangement) that carries the same combinatorial information. Note that we primarily consider non-realizable oriented matroids here.

In an oriented matroid program $(\mathcal{O}, g, f)$, we direct an edge between two adjacent cocircuits X and Y with $X_g = Y_g = +$ as follows: Cocircuit elimination of g between $-X$ and Y yields a unique cocircuit Z and the edge is directed from X to Y (or vice versa, or it stays undirected) if $Z_f = +$ (or $Z_f = -$ or $Z_f = 0$). An oriented matroid program is *Euclidean* iff its cocircuit graph G_f has no directed cycles and an oriented matroid is *Euclidean* iff all its programs are Euclidean. The lexicographic extension $\mathcal{O} \cup p = \mathcal{O}[e_1^{\alpha_1}, \ldots, e_r^{\alpha_r}]$ where $\alpha_i \in \{+, -\}$ for all $1 \leq i \leq r$ is defined as follows, see [1], Proposition 7.2.4: For all cocircuits X in $\mathcal{O}$ holds $X_p = \alpha_i X_{e_i}$ where i is the first index such that $X_{e_i} \neq 0$ and $X_p = 0$ if $X_{e_i} = 0$ for all i. Then, p and e_1 are inseparable elements, which means $X_p = \alpha_1 X_{e_1}$ for all $X_{e_1} \neq 0$ and $X_p \neq 0$. We call a *mutation* M of an oriented matroid a simplicial region in the corresponding pseudosphere arrangement, see [1], Chapter 7.3. Let $M = [e_1, \ldots e_r]$ where the e_i are adjacent to M.

2 Main theorem

While in general inseparable extensions of oriented matroids do not preserve Euclideaness, for the lexicographic extension holds the following theorem:

Theorem 1 (Hochstättler/ Wilhelmi, 2025) *Lexicographic extensions preserve Euclideaness.*

This is [4], Theorem 1.2, the main result of that paper.

3 An application of Theorem 1

With Theorem 1 the following can be shown:

Theorem 2 ([9] Theorem 13) *Each element in a Euclidean oriented matroid of rank and corank ≥ 3 (without coloops) has at least 3 adjacent mutations.*

Proof. [Sketch] We find (mostly, we omit other cases here) to each element f another separable element g. Then $(\mathcal{O}, g, f)$ is Euclidean and the subgraphs G^+ resp. G^- of G_f of cocircuits having $g = +$ and $f = +$ resp. $f = -$ are not empty. We have no directed cycles in

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G^+, hence there must be a source X in G^+ which is a cocircuit of a mutation M_1 adjacent to f (see also [7], Theorem (VI), III, page 318). Analogously, we obtain a mutation M_2 adjacent to f with a cocircuit of G^-. Let now $\mathcal{O} \cup f' = \mathcal{O}[f, \ldots]$ be a lexicographic extension of $\mathcal{O}$. Then f' intersects the two mutations M_1 and M_2 and Theorem 1 yields Euclideaness of $(\mathcal{O} \cup f', f', f)$. There must be a mutation M_3 adjacent to f but not to f', hence different to M_1 or M_2. □

4 Preservation of Euclideaness by mutation-flips

In uniform OMs each mutation corresponds to a mutation-flip, we can 'flip' the mutation and get a new oriented matroid called a *mutant* of the old one, see [1], Theorem 7.3.9.

Theorem 3 ([9], Lemma 21) *Let $(\mathcal{O}, g, f)$ be a Euclidean uniform oriented matroid program having a mutation M adjacent to f but not to g. Then $(\mathcal{O}', g, f)$, where $\mathcal{O}'$ is obtained from $\mathcal{O}$ by flipping of the mutation M, remains Euclidean*

Proof. [Sketch] The cocircuits in $\mathcal{O}'$ having $g = 0$ are the same like in $\mathcal{O}$. Hence the directions of the edeges between two cocircuits outside of the mutation M do not change. It remains to show that cocircuits adjacent to the mutation M can never be part of a directed cycle. This is obvious for all cocircuits X of M having $X_f = 0$ and for the only cocircuit Y of M with $Y_f \neq 0$, all edges from Y to other cocircuits of the graph outside of the mutation must have the same direction. □

5 Applications of both theorems: Mandel OMs

An oriented matroid is *Mandel* iff it has an extension g in general position such that the programs $(\mathcal{O}, g, f)$ are Euclidean for all $f \neq g$. Theorem 1 yields that Euclidean oriented matroids are Mandel. We show:

Theorem 4 ([9] Theorem 17 and 18) *A uniform rank-4 (or minimal non-Euclidean) oriented matroid is Mandel if it has a Euclidean mutant.*

Proof. [sketch] We give here only the construction of the 'Mandel-extension'. The proof of that theorem uses Theorem 3 and Theorem 1, see also [2] for a geometric description. Let $\mathcal{O}_M$ be the Euclidean mutant of $\mathcal{O}$ where the mutation M is flipped and let $f \in M$. Let $\mathcal{O}_{f'} = \mathcal{O}[f, \ldots]$ be a lexicographic extension of $\mathcal{O}$. Then $\mathcal{O}_{f'}$ has a new mutation $M' = M \setminus f \cup f'$. We flip $\mathcal{O}_{f'}$ on the mutation M' and obtain $\mathcal{O}_{f',M'}$. Then f' in $\mathcal{O}_{f',M'}$ is our desired extension. □

This gives us many non-Euclidean but Mandel OMs e.g. the uniform OMs with 8 elements have all a Euclidean mutant. On the other hand, not all OMs are Mandel. It is shown in [7] and in [6] that each element in a Mandel oriented matroid has an adjacent mutation. Three OMs with mutation-free elements are known. The first (with 20 elements) was the $R(20)$ found by Richter-Gebert in 1993, see [8]. Hence not all OMs are Mandel. Furthermore, it holds:

Theorem 5 ([9], Theorem 10) *There is an OM with no mutation-free elements which is not Mandel.*

Proof. [sketch] The $R(20)$ has exactly one mutation-free element f. We extend it lexicographically with an element f' inseparable to f. Then there is a new mutation adjacent to f and to f'. All other mutations stay as they are, hence all elements have adjacent mutations. It is easy to see that the extension cannot be Mandel, because otherwise the $R(20)$ would also be Mandel. □

6 Totally non-Euclideaness

An oriented matroid program is *totally non-Euclidean* iff it has no Euclidean oriented matroid programs. We know from computer help that the $R(20)$ (hence also its dual, see [1], Theorem 10.5.9) is totally non-Euclidean but we will give a proof by hand in an ongoing paper. It would be nice to find other totally non-Euclidean OMs not having these two OMs as a minor. Our final result concerns the mutation-graph (which is the graph having (all or a class of) oriented matroids as vertices where mutants are connected by edges):

Theorem 6 ([9], Corollary 4) *Each path in the mutation-graph of uniform OMs from a Euclidean to a totally Non-Euclidean OM has length ≥ 3.*

Proof. [sketch] Let $\mathcal{O}$ be a Euclidean uniform oriented matroid having two mutations M_1 amd M_2. After flipping M_1, we obtain $\mathcal{O}^1$where the programs with $f \in M_1$ and $g \notin M_1$ (or vice versa) stay Euclidean because of Theorem 3. Then, after flipping M_2 the programs with $f \in M_2 \setminus M_1$ and g in $M_1 \setminus M_2$ (and vice versa) stay Euclidean as well as the programs with $f \in M_1 \cap M_2$ and $g \in M_1^c \cap M_2^c$. One of the two cases must appear. Hence, there are still Euclidean oriented matroid programs after two flips. □

7 Further research

Because the cocircuit graph of a Euclidean OM has no directed cycles, it yields a partial (extended to a linear) ordering of these cocircuits. We showed in [5] that these orderings are always a shelling of the polar (we say a *node-shelling*) of the big face lattice and also a shelling of the Las Vergnas lattice of an oriented matroid, see [1], Chapter 4 for the notions. It is an open question if these two lattices are shellable or not, at least we can show that for Euclidean oriented matroids, they are. In an upcoming paper (joint work with W.Hochstättler) we will show that the $R(20)$ has regions where some programs are still Euclidean. By glueing together different node-shellings of these regions we will show that at least a halfspace of the $R(20)$ has a node-shelling.

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